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ERROR ANALYSIS FOR
GEOTECHNICAL ENGINEERING

by

Gregory Baecher
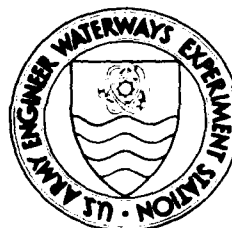
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An introduction to practical techniques of error analysis for geotechnical engineering. The intended audience is the practicing geotechnical engineer with little background in probability or statistics. An analysis procedure is presented by which errors entering geotechnical calculations can be quantified and their effect on predictive accuracy evaluated. The approach involves three phases: (a) identification of potential sources of error, (b) assessment of the magnitude of each source, and (c) determination of the influence of each source on calcuations. A companion report, "Statistical Analysis of Geotechnical Data," provides related introductory material on the statistical analysis of geotechnical site-characterization data.

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PREFACE

This report, prepared by Gregory B. Baecher of NEXUS Associates, Wayland, Massachusetts, with assistance from D. DeGroot, and C. Erikson, under Contract DACW39-83-M-0067, provides details for the statistical analysis of geotechnical engineering aspects of new dam projects. It was part of work done by the US Army Engineer Waterways Experiment Station (WES) in the Civil Works Investigation Study (CWIS) sponsored by the Office, Chief of Engineers, US Army. This study was conducted during the period October 1983 to September 1985 under CWIS Work Unit 32221, entitled "Probabilistic Methods in Soil Mechanics." Mr. Richard Davidson was the OCE Technical Monitor.

This report is an introduction to practical techniques of uncertainty or error analysis for use in geotechnical engineering. The intended audience is the practicing geotechnical engineer with little or no background in probability theory or statistics. A companion report, "Statistical Analysis of Geotehcnical Data" (Contract Report GL-87-1), has been prepared under the same contract. A third report, entitled "Statistical Quality Control of Engineered Embankments" (Contract Report GL-87-2), has also been prepared but is independent in content.

Ms. Mary Ellen Hynes-Griffin, Earthquake Engineering and Geophysics Division (EEGD), Geotechnical Laboratory (GL), WES, was the Contracting Officer's Representative and WES Principal Investigator for CWIS Work Unit 32221. General supervision was provided by Dr. A. G. Franklin, Chief, EEGD, and Dr. W. F. Marcuson III, Chief, GL.

COL Dwayne G. Lee, CE, was Commander and Director of WES during the publication of this report. Dr. Robert W. Whalin was Technical Director.

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PART I: INTRODUCTION

Background

Good engineering practice requires that the possible magnitude and direction of errors in engineering performance calculations be systematicaly evaluated. Traditionally, geotechnical engineers have used sensitivity analysis to perform such evaluation. However, advances in geotechnical testing and modeling allow benefits to be taken of the more advanced, quantitative approaches to error analysis common in other branches of engineering. These techniques provide an explicit procedure for assessing uncertainty and error in engineering predictions and are easily tailored to the needs of geotechnical engineering practice.

Purpose

The purpose of this report is to provide potential users of error analysis methods in geotechnical engineering with a practical introduction to concepts, definitions, and techniques. The report is not exhaustive. The intent of the report is to provide sufficient detail to allow a reader already versed in geotechnical engineering but not in probability or statistics to undertake error analysis for routine problems encountered in geotechnical engineering practice. This report complements materials presented in "Statistical Analysis of Geotechnical Data," Contract Report GL-87-1, and presumes familiarity with statistical representations of soil parameter estima es up to the level of that companion report.

General Description of Error Analysis

The general approach to error analysis involves three phases: (a) identification of possible sources of error, (b) assessment of the magnitude of error possibly contributed by each source, and (c) determination of the influence of each source of error on calculated results. The product of an error analysis is a quantified statement of the confidence to be placed on predictions of a structure's performance which result from engineering calculations. This statement of confidence represents the cumulative effect of the uncertainties inherent to data collection and interpretation, and engineering modeling.

Report Organization

This report is organized in five sections. Following the introduction, Section II describes the sourses of error and uncertainty in geotechnical analysis. Part III presents mathematical techniques for calculating the effects of those errors and uncertainties on predictions of facility performance. Part IV introduces the concept of a reliability index to provide a single measure of safety incorporating both the best estimate of factor of safety and the potential error in the best estimate. Finally, Part V discusses the use of risk analysis in making geotechnical engineering design decisions.

PART II: SOURCES OF ERROR IN GEOTECHNICAL ANALYSIS[†]

Many uncertainties affect geotechnical predictions. Some can be quantified, some not. In an approximate way these uncertainties can be divided into five main groups:

- ¶ Site conditions,
- ¶ Loads,
- ¶ Model inaccuracies,
- ¶ Construction and quality control problems, and
- ¶ Omissions and gross errors.

The most important for engineering <u>analysis</u> are the first three. Site condition, loads, and models appear in calculations, and the uncertainties associated with them are quantifiable. In contrast, uncertainties caused by construction or quality control problems and by omissions or errors do not appear in calculations, and they are seldom quantifiable. These uncertainties are accomodated in other ways, as for example, by quality assurance or by design checking.

Error in Geotechnical Calculations

If attention is restricted to geotechnical aspects of calculations, the principal uncertainties that must be dealt with are site conditions and geotechnical models, and a further and more specific subdivision of sources of errors is possible. This leads to four sources which are the focus of error analysis:

1. Soil variability.
2. Measurement noise.
3. Measurement and model bias.
4. Statistical error due to limited measurements.

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† Part II summarizes background materials from the companion report, "Statistical Analysis of Geotechnical Data," (Contract Report GL-87-1). Readers conversant with that report may wish to skip forward to Part III. Readers desiring further discussion of the material in Part II are referred to the companion report.

These are the sources of uncertainty which affect calculated predictions. The first two, soil variability and measurement noise, appear as data scatter (Fig. 1). The latter two, measurement or model bias and statistical error, cause systematic error in predictions.

Data Scatter

The scatter among geotechnical measurements is usually large. This scatter reflects two things: (a) real variability within a soil deposit and (b) random measurement error or 'noise'. A major benefit of statistical analysis is the ability to separate real variability from noise, and thereby lessen the magnitude of data scatter and reduce uncertainty.

In any real soil deposit, engineering properties vary somewhat from point to point, even within a stratum or zone that otherwise appears uniform. This real or spatial variation leads to differences in the way individual elements of a soil deposit behave. Unlike differences in soil properties among strata or zones, this form of spatial variation is not usually accounted for deterministically in a soil model. As a result, it causes uncertainty in predictions of how individual elements of the soil mass perform.

Random measurement error in contrast to real soil variability is extraneous 'noise' in soil property measurements. It is caused by operator or instrumental variations from one test to the next. It does not reflect real variations in soil properties, except possibly on a very small scale (e.g., shell fragments or pebbles that distort test results). Nevertheless, unless steps are taken to assess the magnitude and importance of random measurement error, it cannot be distinguished from real variation in soil properties.

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Thus, noise increases the uncertainty in soil properties estimates and leads to greater conservatism in design than would otherwise be necessary.

Systematic Error

Systematic error is bias. If a systematic error of, say, 10% in some soil property is made at one location, the same 10% error is made everywhere. The distinction between spatial variation and systematic error is important. The uncertainties caused by these two sources affect engineering calculations in different ways. For example, uncertainties due to spatial variation lead to a high likelihood of some fraction of a facility performing inadequately. Uncertainties due to systematic error, on the other hand, lead to a small likelihood of the entire structure performing inadequately.

In soil property estimation, systematic error is caused in two ways: (a) bias in measurement techniques and the models used to interpret measurements, and (b) statistical estimation error. Measurement and model bias is common in geotechnical engineering. It is caused by such things as soil disturbance, or by a difference between how a property is measured and how a structure imposes loads, or by simplifications in how soil behavior is modelled. Statistical bias is also common. It is caused by limited numbers of data.

Combining The Sources of Uncertainty

Together, data scatter and systematic error constitute the major uncertainties of geotechnical calculations. However, the effects of these components differ, as do the way each propagates through an engineering model. The most important concept of error analysis has nothing to do with mathematics, it has to do with separating these sources of uncertainty.

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Error analysis is based on, (a) separating the sources of error in geotechnical predictions, (b) analyzing the individual effect of each source of error on an engineering prediction, and (c) combining the effects of each source of error to obtain the overall reliability of a prediction. Error analysis treats calculations and engineering models. It is not 'probabilistic design;' it is more akin to quality control. The result of error analysis is a reliability index which summarizes the overall confidence that can reasonably be placed in an engineering calculation.

Fig. 2 shows how the procedure is used. At the beginning, statistical methods are used to estimate the four components of uncertainty from laboratory tests, field data, and various geotechnical considerations. These components are combined in a design profile used as input for modeling. In contrast to traditional methods, this design profile is not a conservative estimate of soil properties, but a 'best' estimate. Uncertainty is accomodated by specifying standard deviation envelopes on the profile. The best estimate profile is used as input to the geotechnical model to obtain a best estimate of facility performance, for example, a best estimate of factor of safety, F. The envelopes expressed as standard deviations are propogated through the model to obtain a corresponding envelope or standard deviation on the prediction, e.g., a standard deviation on F. The mean and standard deviation of the prediction are combined in a so-called reliability index (Part IV) to provide a measure of confidence.

Describing Uncertainty

Assessments of soil properties for most purposes are adequately expressed by two numbers, (a) a best estimate, and (b) a measure of uncertainty. The

average value and the standard deviation, respectively, are used to express these two attributes. When more than one property is estimated, another attribute becomes important. This is the association between the uncertainties in different parameter estimates. The correlation coefficient is used to express this association.

Average = 'Best Estimate'

The average or mean of a set of measurements $\underline{x} = \{x_1,...,x_n\}$ is denoted m_x and defined as,

$$m_x = \frac{1}{n} \Sigma\ x_i = \text{"mean"} \qquad . \qquad (1)$$

In effect, the mean is the center of gravity of the measurements along the x-axis. It is used as the best single-valued estimate of x.

The compaction control measurements shown in Fig. 3 show water content and density data. In Fig. 4 the same data are displayed as histograms, showing the number of measurements falling within specified intervals. The mean of the water content data using Eqn. 1 is $m_x = 0.45\%$; the mean of the compaction data is $m_x = 98.7\%$.

Standard Deviation = 'Uncertainty'

The standard deviation of the measurements $\underline{x}$ is their variation with respect to the mean, expressed as the square root of the sum-of-squared variations,

$$s_x = \sqrt{\frac{1}{n-1} \Sigma\ (x_i - m_x)^2} = \text{"standard deviation"} \qquad (2)$$

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In effect, the standard deviation is the root of the moment of inertia of the data about the mean. s_x measures the dispersion or uncertainty about the value of x. The standard deviation of the data in Fig. 3 is calculated by Eqn. 2 to be s_x=1.3% for the water content data and s_x=2.6% for the compaction data.

The proportional uncertainty or standard deviation normalized by the mean is called the coefficient of variation and denoted Ω_x,

$$\Omega_x = s_x/m_x = \text{"coefficient of variation"} \quad . \tag{3}$$

The coefficients of variation of the data in Fig. 3 are Ω_x=1.3%/0.45%=2.89 for water content, and Ω_x=2.6%/98.7%=0.03 for compacted density.

Just as in mechanics it is often convenient to deal with the moment of inertia, rather than its square root, so, too, in analyzing uncertainty it is often convenient to deal with the square of the standard deviation rather than s_x itself. The square of the standard deviation is called the variance,

$$V_x = s_x^2 = \text{"variance"} \quad . \tag{4}$$

Given the similarity of equations 1 and 2 to mechanical moments, the mean and variance are often called the first and second (statistical) moments of the uncertainty in x.

Correlation Coefficient

When dealing with two or more soil properties, uncertainties in estimates may be associated with one another. That is, the uncertainty in one property estimate may not be independent of the uncertainty in the other estimate.

Consider the problem of estimating 'cohesion' and 'friction' parameters of a Mohr-Coulomb strength envelope. If the slope of the envelope to a set of Mohr circles is mistakenly estimated too steeply, then for the line to fit the bulk of the data the intercept must be made too low. The reverse is true if the slope is estimated too flat. Thus, uncertainties about the slope and about the intercept are not independent, they relate to one another.

The correlation coefficient for paired data $\underline{x},\underline{y} = \{ (x_1,y_1), ..., (x_n,y_n) \}$ is denoted r_{xy}, and defined as,

$$r_{xy} = \frac{1}{n} \Sigma \left(\frac{x_i - m_x}{s_x}\right) \left(\frac{y_i - m_y}{s_y}\right) = \text{"correlation coefficient"} \quad . \quad (5)$$

In effect, the correlation coefficient is equivalent to a normalized product moment of inertia in solid mechanics. It expresses the degree to which two parameters vary together. The correlation coefficient is non-dimensional because deviations of x and y from their respective means are measured in units of the respective standard deviations.

The value of r_{xy} may vary from +1 to -1. $r_{xy}=+1$ implies a strict linear relation with a positive slope, perfect correlation. $r_{xy}=-1$ implies a strict linear relation with a negative slope, perfect inverse correlation. $r_{xy}=0$ implies no association at all, independence. Fig. 5 shows scatter plots with various r_{xy} values.

The corresponding dimensional form of Eqn 5, that is, using the absolute deviations of x and y rather than normalized deviations, is called the covariance and denoted,

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$$C_{xy} = \frac{1}{n-2} \Sigma \ (x_i - m_x)(y_i - m_y) = \text{"covariance"} \quad . \quad (6)$$

From the definitions of Eqns. 5 and 6,

$$r_{xy} = \frac{C_{xy}}{s_x s_y} \quad . \quad (7)$$

Autocovariance

Thus far, the fact that soil properties are spatial variable has been ignored. Soil properties have not only magnitude but also location. The spatial quality of soils data has important implications for it both strongly affects engineering predictions and increases the amount of information that can be squeezed from a testing program. The salient aspects of spatial variability from an error analysis view are analyzed using a statistical concept called autocovariance.

In an approximate way, spatial variability of data can be summarized by two measures: the variance of the data about their mean, and the waviness or frequency content of the variability in space (Fig. 6). The longer the period of this 'waviness' the further the data may be spatially extrapolated. Autocorrelation is used to measure 'waviness.'

Autocovariance measures the statistical association between data of the same type made at separate locations. For example, the properties of two adjacent soil elements tend to be similar. If one is above average, the other tends to be above average, too. They are associated. Conversely, the

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properties of widely separated elements are not necessarily similar. If one is above average, the other may or may not be. They are not associated. This association of properties in space can be measured by the correlation coefficient of Eqn. 5 and is called autocorrelation because the data are all of the same type.

For data x_i, where i = the location of the measurement, the autocorrelation of data separated by interval δ is,

$$R_x(\delta) = \frac{1}{(n_\delta - 1)V_x} \sum_{i=1}^{n_\delta} (x_i - m_x)(x_{i+\delta} - m_x) \quad , \tag{8}$$

the sum taken over all pairs of data having separation δ, their number being n_δ. Eqn. 8 applies to the case in which the mean of x is constant in space. More general cases are considered in the companion report, "Statistical Analysis of Geotechnical Data" (Contract Report GL-87-1). Autocovariance is related to autocorrelation as covariance is to correlation. The autocovaraiance of data at points separated by a distance δ is,

$$C_x(\delta) = \left(\frac{1}{n\delta - 1}\right) \sum_{i=1}^{n_\delta} (x_i - m_x)(x_{i+\delta} - m_x) \quad , \tag{9}$$

Autocorrelation expressed a function of separation distance δ is said to be the autocorrelation function, and autocovariance expressed as a function of separation distance is said to be the autocovariance function.

Figure 7 shows the autocorrelation function for standard penetration test blow count data measured in a silty hydraulic fill. For convenience, the autocorrelation or autocovariance function is often indexed by the distance at

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which it decays to 1/e of its original value, in which e is the base of the natural logarithms. For Fig. 7, this distance is about 100 feet.

Estimating Uncertainty

This section considers specific procedures for quantifying the uncertainties identified above. More detail is provided in the report, "Statistical Analysis of Geotechnical Data" (Contract Report GL-87-1).

Data Scatter

The scatter in soil data reflects two things (spatial variability and noise) but it is measured by a single parameter, the data variance, V_x. Therefore, it is not possible to separate soil variability from measurement error simply by inspection; another approach to estimating the fraction of data scatter contributed by either of these two sources must be used. The most convenient and accurate is through the autocovariance function. The autocovariance function reflects the spatial structure of variability in a set of data. This structure differs depending on how the scatter is divided between spatial variability and measurement noise. Each component has a characteristic signature in the autocovariance function.

As a good approximation, measurements taken in the laboratory or field can be modelled as,

$$z = x + e , \tag{10}$$

in which z is the measurement, x is the real soil property, and e is random measurement error. After some algebra, the autocovariance function of the set of measurements turns out to be related to the autocovariance functions of x and e by,

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$$C_z(\delta) = C_x(\delta) + C_e(\delta) \quad . \tag{11}$$

The autocovariance of x equals V_x at $\delta=0$, and approaches 0 as δ increases. The autocovariance of e, on the other hand, equals V_e at $\delta=0$, but equals 0 for any $\delta \neq 0$, that is, it is a spike. This is due to the assumption--and empirical observation--that noise is independent from one test to another. Thus, for $\delta \neq 0$ the covariance of the e's is zero. Therefore, by extrapolating the observed autocovariance function back to the origin, an estimate of V_x and V_e is obtained. This is shown in Fig. 8. In this case the variance of the measurement noise is about 50% of the data scatter. For other in situ measurements and other soils, measurement error variances have been found to contribute anywhere from 0 to as much as 70% of the data scatter (e.g., Baecher, Marr, Lin, and Consla, 1983).

Systematic Error

The principal sources of systematic error are statistical estimation error, and measurement or model bias.

Systematic error due to statistical estimation of soil parameters is calculated from statistical theory. The most significant of these errors is in the mean soil property. As an approximation, although a robust one, the variance of the statistical error in this mean is,

$$V_{m_x} = V_z/n \quad , \tag{12}$$

in which n is the number of measurements. Note, although random measurement error can be eliminated from the data scatter variance to yield a reduced

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uncertainty, it does contribute to statistical error. Its effect on statistical error can only be lessened by making more measurements.

Statistical errors in the estimates of other parameters, for example, the variance of soil properties, also exist and can be readily calculated; however, in most cases they have only a second-order effect on predictions and may be safely ignored.

The last of the major sources of uncertainty, measurement and model bias, is the most difficult to estimate. Usually, the only way to do so is by comparison of predicted with observed performance or by field-scale experiments. This has been done by Bjerrum (1972) for field vane strengths of normally consolidated clay, and has been attempted by other workers for other measurements. Such an approach aggregates a large number of uncertainties or biases together, including those due to inaccuracies of theory and method of analysis. In other endeavors, such as assessing nuclear site safety, measurement and model bias are sometimes subsumed under the name, model validation.

In the case of Bjerrum's work, the joint effect of bias in the field vane procedure and bias in 2D stability analysis based on modified Bishop method leads to the correction factor shown in Fig. 9. The best estimate of this correction factor, μ, as a function of plasticity index is its mean, m_μ, given plasticity index. The variation of back-calculated μ's about the mean is summarized in a variance V_μ which expresses the uncertainty in knowledge about the bias term.

Autocovariance

This section considers a simple and often used approach to estimating autocovariance, the moment estimate. More detailed discussion of the

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statistical aspects of estimating autocovariance is presented in DeGroot (1985)

and in "Statistical Analysis of Geotechnical Data" (Contract Report GL-87-1).

Consider the case of measurements at equally spaced intervals along a

line, as for example in a boring. Presume that the measurements $\underline{x}$ =

$\{x_1,...,x_n\}$ contain no measurement error. The observed autocovariance of the

measurements at separation δ is,

$$\hat{C}_x(\delta) = \frac{1}{n_\delta-1} \ \Sigma \ (x_i-m_x)(x_{i+\delta}-m_x) \tag{13}$$

in which n_δ = the number of pairs of data at separation distance δ and the mean

m_x is assumed spatially constant. This is called the sample autocovariance.

The sample autocovariance is used to estimate the real autocovariance $C_x(\delta)$ for

separation distance δ.

In the general case the measurements are seldom uniformly spaced and, at

least in the horizontal plane, seldom lie on a line. For such situations Eqn.

13 can still be used, but with some alteration. The most common way to

accomodate nonuniformly placed measurements is by dividing separation distances

into bands, and then taking the averages of Eqn. 13 within those bands. This

introduce some bias into the estimate but for most engineering purposes it is

sufficiently accurate.

Design Profile

The total uncertainty in engineering properties at a point in the soil

profile reflects the combination of data scatter and systematic error. For

modeling purposes it is convenient to draw a design profile of soil properties

vs. depth. About this profile are drawn two sets of standard deviation

envelopes. One set describes point to point variability. The other set

describes uncertainty in the mean.

$$\begin{pmatrix} \text{uncertainty} \\ \text{in x} \end{pmatrix} = \begin{pmatrix} \text{soil} \\ \text{variability} \end{pmatrix} + \begin{pmatrix} \text{statistical} \\ \text{uncertainty} \end{pmatrix} + \begin{pmatrix} \text{measurement} \\ \text{uncertainty} \end{pmatrix} \quad (14)$$

$$= \quad \begin{matrix} \text{spatial} \\ \text{variability} \end{matrix} \quad + \quad \begin{matrix} \text{uncertainty} \\ \text{in mean} \end{matrix} \quad (15)$$

As discussed in Part III, the contributions to uncertainty in a soil property x add together through their respective variances. That is, the variance of x is found by summing the variance of spatial variability and the variance in the mean,

$$V_{x,\text{total}} = V_{x,\text{spatial}} + V_{m_x} \quad , \quad (16)$$

in which $V_{x,\text{total}}$ = the total variance of the soil property x, estimated at a point location, $V_{x,\text{spatial}}$ = the variance due to spatial variability of the soil property, and V_{m_x} = the variance due to uncertainty in the mean of x. The variance due to spatial variability is

$$V_{x,\text{spatial}} = V_z - V_e \quad (17)$$

in which V_z = the variance of the data scatter (i.e., measurements,) and V_e = the variance of the measurement noise. The variance of the uncertainty in the mean of x is,

$$V_{m_x} = V_{x,\text{statistical}} + V_B \quad , \quad (18)$$

in which $V_{x,\text{statistical}}$ = the variance due to statistical error in the mean, and V_B = the variance due to measurement bias.

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Since the contribution of random measurement error appears only in its effect on statistical error, this means that V_x in specific instances can be considerably less than the data scatter variance V_z.

Returning to the measurement model of Eqn. 10, but applying a measurement bias correction (i.e., calibration) factor B,

$$z = Bx + e \qquad . \qquad (20)$$

The factor B is an unknown constant that influences the measurement of the actual soil property x. As developed in the report "Statistical Analysis of Geotechnical Data" (Contract Report GL-87-1), the variance of z is related to the variances of B, x, and e by

$$V_z = m_B^2 V_x + m_x^2 V_B + V_e \qquad (21)$$

Solving for the variance of x

$$V_x = (1/m_B^2) (V_z - V_e) + m_x^2 V_B/m_B^2 \qquad , \qquad (22)$$

$$= (1/m_B^2) (V_z - V_e) + m_x^2 \Omega_B^2$$

in which V_B is the uncertainty in the appropriate value of the bias correction B. For example, for field vane data, V_B is found from the scatter of calibration data of the type compiled by Bjerrum (1962). The first term in the right-hand-side (RHS) of Eqn. 22 is the contribution of spatial variation of soil properties to uncertainty in x; the second term is the contribution of uncertainty about B to uncertainty in x. Since uncertainty about the proper

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value of B always increases the uncertainty in an estimate, whether in going from x to z or z to x, its contribution is always positive.

The effect of statistical error in the mean m_x, due to limited numbers of measurements, adds directly to the RHS of Eqn. 22, as suggested by Eqns. 16 and 18. From Eqn. 12 the variance of the statistical error in m_x is,

$$Vm_x = V_z/n \approx (V_x + V_e)/n \quad . \tag{23}$$

Note that, although measurement noise can be removed from the data scatter, it still retains an effect on statistical uncertainty

The overall error in the estimate of soil properties at any point in the soil mass is found by combining the individual contributions of soil variability, measurement bias and statistical error to obtain,

$$\left(\begin{array}{c} \text{uncertainty} \\ \text{in soil} \\ \text{property x} \end{array}\right) = \left(\begin{array}{c} \text{spatial} \\ \text{variability} \end{array}\right) + \left(\begin{array}{c} \text{measurement} \\ \text{bias} \\ \text{uncertainty} \end{array}\right) + \left(\begin{array}{c} \text{statistical} \\ \text{uncertainty} \end{array}\right) \tag{24}$$

$$V_x \doteq (1/m_B^2)(V_z - V_e) + m_x^2 \Omega_B^2 + V_z/n \tag{25}$$

The first term on the RHS of Eqn. 25 is the uncertainty caused by scatter of the data about a trend in space. The combined effect of the second and third terms is the uncertainty on the trend itself.

Example: Field Vane Strength Data

The field vane (FV) strength data of Fig. 10 were collected in 40 borings along the axis of a proposed embankment. The strength profile for end of

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construction (i.e., undrained) conditions was estimated on the basis of the field vane data. From visual inspection the mean field vane strength in the marine clay appears approximately constant with depth, below a weathered crust. Data classified as in the crust were treated separately from lower data, while high strength outliers within the non-crust material were eliminated.

The mean and variance of the FV data in the marine clay are shown in Table 1. The coefficient of variation of 20 to 30% is large although not exceptionally so. Analysis of the spatial structure variability about a constant mean yields estimates of the vertical and horizontal correlation distances for the marine clay of about 1m and 30m, Figs. 11 and 12. Extrapolating the autocovariances back to the origin yields an estimate of 20kPa^2 for the noise (Fig 13). Note, to the extent that the apparent spike at r=0 is due to small scale variability, rather than noise, the estimated "measurement error" from the vertical and horizontal directions need not be the same. However, in the present case they appear to be.

The statistical error is approximately,

$$V_{m_x} = V_z/40 = 1.66 \text{ kPa}^2 \quad , \tag{19}$$

which assumes tests to be independent. Given the separation of the tests relative to the autocovariance distances, this assumption seems satisfactory.

Measurement bias for the profile derives from errors introduced by the FV device and simplified Bishop analysis. For this purpose, Bjerrum's (1972) FV correction factor, μ, was used (Fig. 9). Bjerrum's μ is found by comparing measured undrained strengths using the FV with undrained strengths

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backcalculated from observed slope failures. For the marine clay (PI=20) at the site, $m\mu$=1.0 and the uncertainty in μ reflecting the scatter about the best fitting curve of Fig. 9 was estimated using regression analysis to be $\Omega\mu$=0.075.

Thus, having estimated each of the component variances of Eqn. 14 the design profile can be developed. The average undrained strength with depth is approximately constant, $m_u \approx$ 34.5 kPa, and the variance is found by Eqn. 14. The result is shown in Fig. 14. The innermost envelopes show ± one standard deviation of the systematic uncertainty (i.e., error in the mean). The outer envelopes show ± one standard deviation of total uncertainty for soil properties at a point. The total uncertainty is the sum of spatial variation plus uncertainty about the mean, calculated using Eqn. 16.

Example: SPT Blow Count Data

The boring program at the site of a low earth dam consisted of approximately 63 standard penetration test borings located along the dam axis and along a section perpendicular to the axis at the spillway location (Fig. 15). Additional borings were concentrated in a zone along the axis between stations 27+50 and 37+00 where deep solution activity was discovered in the limestone foundation.

An interpreted profile along the axis is shown in Figure. 16. The soil conditions along the axis can be roughtly divided into three sections on the basis of SPT blow counts: Station 4+00 to 13+00, 17+00 to 24+50, and 25+00 to 32+00. Blow count data for the three sections are shown in Figs. 17, 18, and 19; and summarized in Table 2.

The average blow counts increase from low to high station across the dam axis. The soils underlying the right wing of the dam have very low N values.

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The blow count data in each section closely follow a Normal distribution of relative frequency. This trend is exhibited in probability-grid plots, in which Normal distributions of frequency plot as lines, but is less obvious simply by inspection of the corresponding histograms. The coefficients of variation in each section are approximately constant at 50%. The blow count data were not corrected for overburden.

The spatial structure of the blow count data was investigated directly on the raw blow counts and on detrended blow counts. The autocovariance function for the non-detrended average blow counts in each boring is shown in Figure 20, estimated using Eqn. 13.

The upper plot shows the mean estimates calculated as per Equation 19. The lower plot shows variation about the mean estimate, represented as:

$\leftarrow$ maximum

$\leftarrow$ 75th fractile

$\leftarrow$ mean
$\leftarrow$ median

$\leftarrow$ 25th fractile

$\leftarrow$ minimum
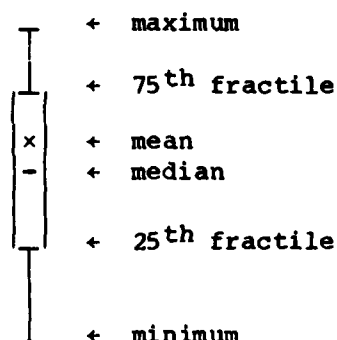
the value of $C_N(\delta)$ at $\delta=0$ is the variance of the boring-averaged blow count data across the site. From Fig. 20 the variance of average blow counts in each boring is about 6.0 bpf^2, and and the autocorrelation distance is about 350'.

A corresponding autocovariace function for individual data at elevation 660 is shown in Figure 21. The variance of these data is 12.5 bpf^2, and the autocorrelation distance is about 200 to 250'. The strong spatial structure in both cases is due to the trend of blowcounts along the embankment axis.

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Measurement noise in the data was estimated using the autocovariance function. By extrapolating the sample autocovariance of z back to the origin at $\delta=0$ an estimate of both V_x and V_e is made. An interesting finding from the blow count data at the dam site is that the measurement error in the data appears to be very small. This is surprising. SPT blow counts are widely thought to be noisy measurements and have been shown elsewhere to have noise components in excess of 50%. The small measurement noise in the present data can be inferred from the autocovariannce function. There is essentially no discernable spike at $\delta=0$, and thus the variance of the measurement error V_e would appear to be about zero.

The reason that measurement noise in these SPT data appears to be so low may have to do with the looseness of the alluvium and the consequent low average blow counts. A more detailed discussion of the statistical analysis of these data is presented in the report "Uncertainty Analyses for Dam Projects," Final Report, Contract Report GL-87-4.

Systematic error in the blow count profile is normally caused both by measurement bias and by statistical uncertainty. In the present case involving blow count data there was little reason to suspect serious measurement bias. The data are used directly in constructing a soil property profile, rather than being translated into a more fundamental parameter such as strength or deformability. Therefore, there is no transformation by which bias is introduced. While field operations themselves may introduce bias in SPT data, there was no reason to suspect such bias in the present case. As a result, they were neglected.

Statistical error on the mean was estimated using Eqn. 12. Typically there were about 20 blow count measurements at any one depth, given that the

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axis was divided into three zones. Obviously, by dividing the site into zones a closer fit to the mean properties in each zone can be made, but at the same time the number of data used to estimate the means goes down.

The results of the analysis of the blow count data are shown in Figures 22, 23, and 24, in which the mean profile and standard deviation envelopes are denoted as:

```
                                    mean
                                     ↓
 -standard deviation of mean      _____        +standard deviation of mean
 mean plus spatial variation |----|__x__|----|   mean plus spatial variation
                                  ↑      ↑
                     ± standard deviation of mean
```
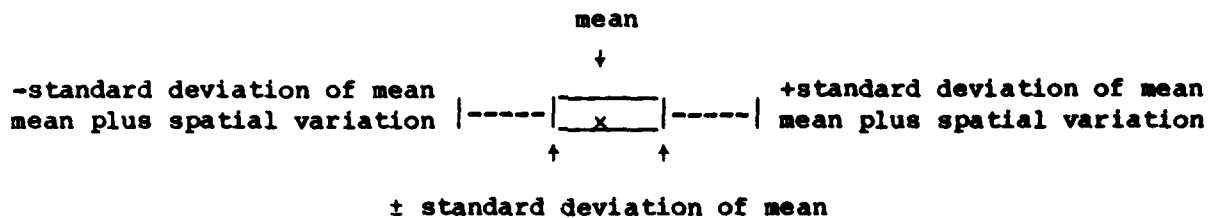
The inner envelopes are ± one standard deviation of the uncertainty in the mean and the outer envelopes are ± one standard deviation of the error in the mean plus the spatial variation. Uncertainty in groundwater level is denoted by a mean and ± one standard deviation of the spatial variation.

Use of the Design Profile

The design profile provides a means for summarizing the magnitudes of potential error in the soil properties that are used as input to engineering analyses. Importantly, it separates systematic errors in the mean from spatial variations one location to another. In the next part of this report, the design profile is used as quantitative input to a geotechnical analysis--for example, a slope stability calculation, or a settlement calculation--in order to assess the effect of uncertainty on predicted factors of safety or another prediction. In that assessment, the importance of systematic errors and importance of spatial variations are very different.

Table 1. Component Variances for Field Vane Data in the Marine Clay.

Component Variances for Undrained Strength Profile		
Variance Component	Marine Clay	Lacustrine Clay
Mean, m_X	34.5 kPa	31.2 kPa
Variance Components:		
Spatial, V_X	39.9	72.0
Measurement noise, V_e	26.4	0
Data scatter total, V_z	66.3 kPa^2	72.0
Statistical error, V_z/n	1.7	2.0
Measurement/Model Bias, V_B	5.3	21.9
Total Variance, $V_{X,total}$ (point-to-point)	83.3 kPa^2	95.9 kPa^2

Table 2. SPT Summary Statistics for Carters Project Reregulation
Dam.

SPT SUMMARY STATISTICS FOR REREGULATION DAM FOUNDATION

section	mean (bpf)	standard deviation (bpf)	coefficient of variation	length (feet)
Station 4+00 to 13+00	4.8	2.9	0.60	900
Station 17+00 to 24+50	6.9	2.8	0.41	750
Station 25+00 to 32+00	8.9	4.4	0.49	700

Table 3. Component Variances for Factor of Safety Calculation in Embankment Stability Analysis.

Principal Uncertainties in Stability Calculation

Parameter	Symbol	Variance		
		Systematic	Spatial	TOTAL
Effective friction angle of fill	ϕ'	3.0		4.0 $\circ^2$
Density of fill	γ_{FILL}	1.0		2.0 tcm^2
Depth of dessicated crust	D_{crust}	0.036		1.0 m^2
Depth to till	D_{till}	1.0		1.0 m^2
Undrained strength of lacustrine clay	$c_u(L)$	24.9		99.7 kPa^2
Undrained strength of marine clay	$c_u(M)$	7.6		47.6 kPa^2
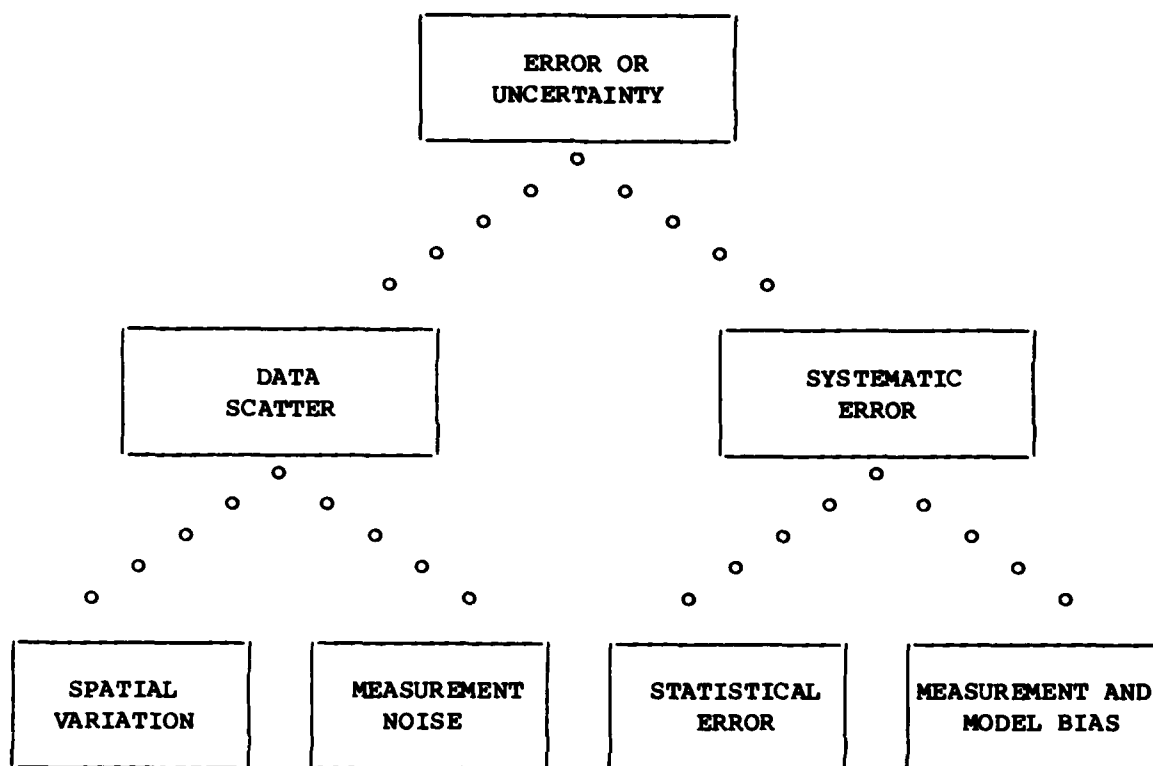
Figure 1 -- Sources of Error or Uncertainty
in Soil Property Estimates.

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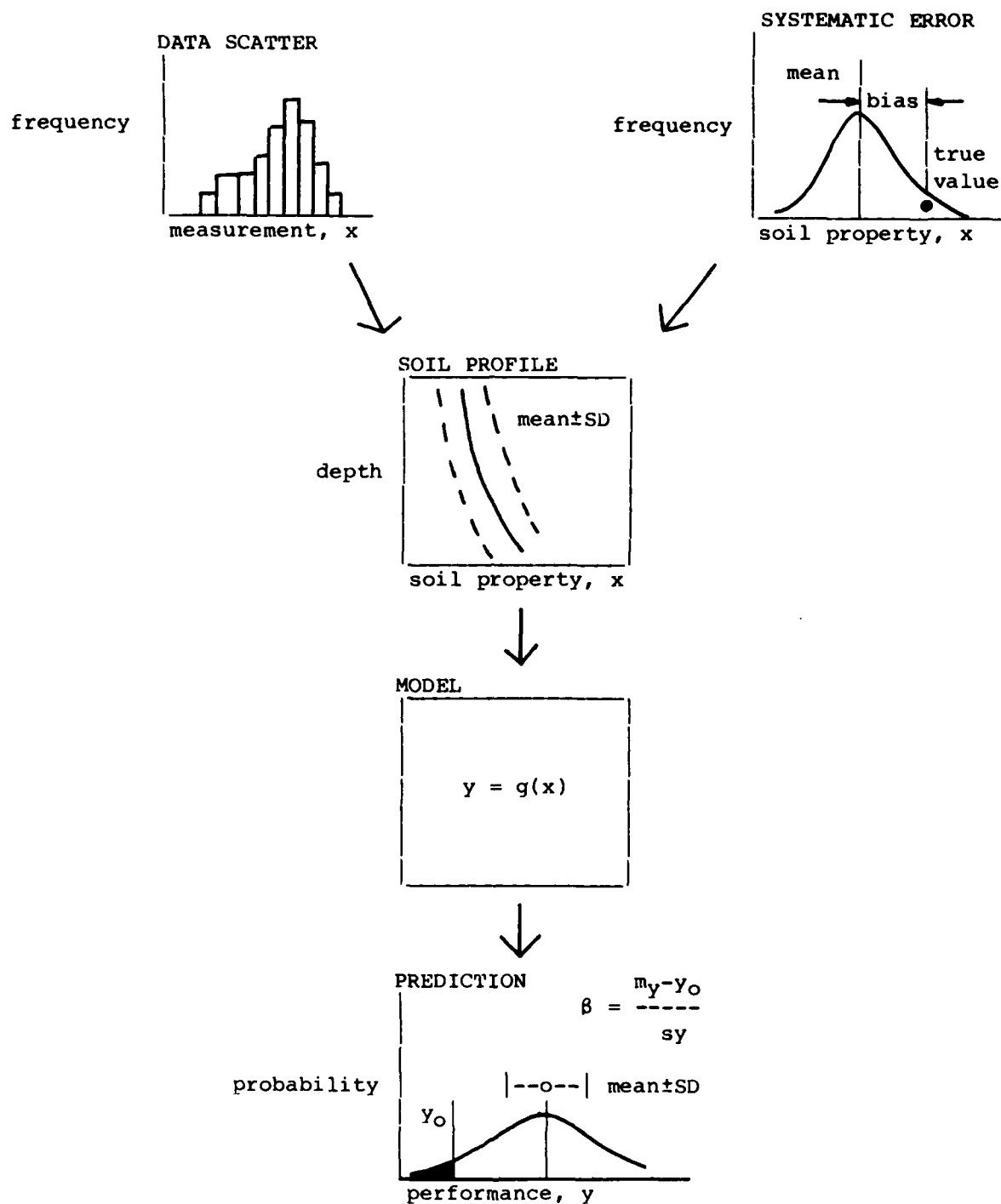
Figure 2 -- Error Analysis Procedure.
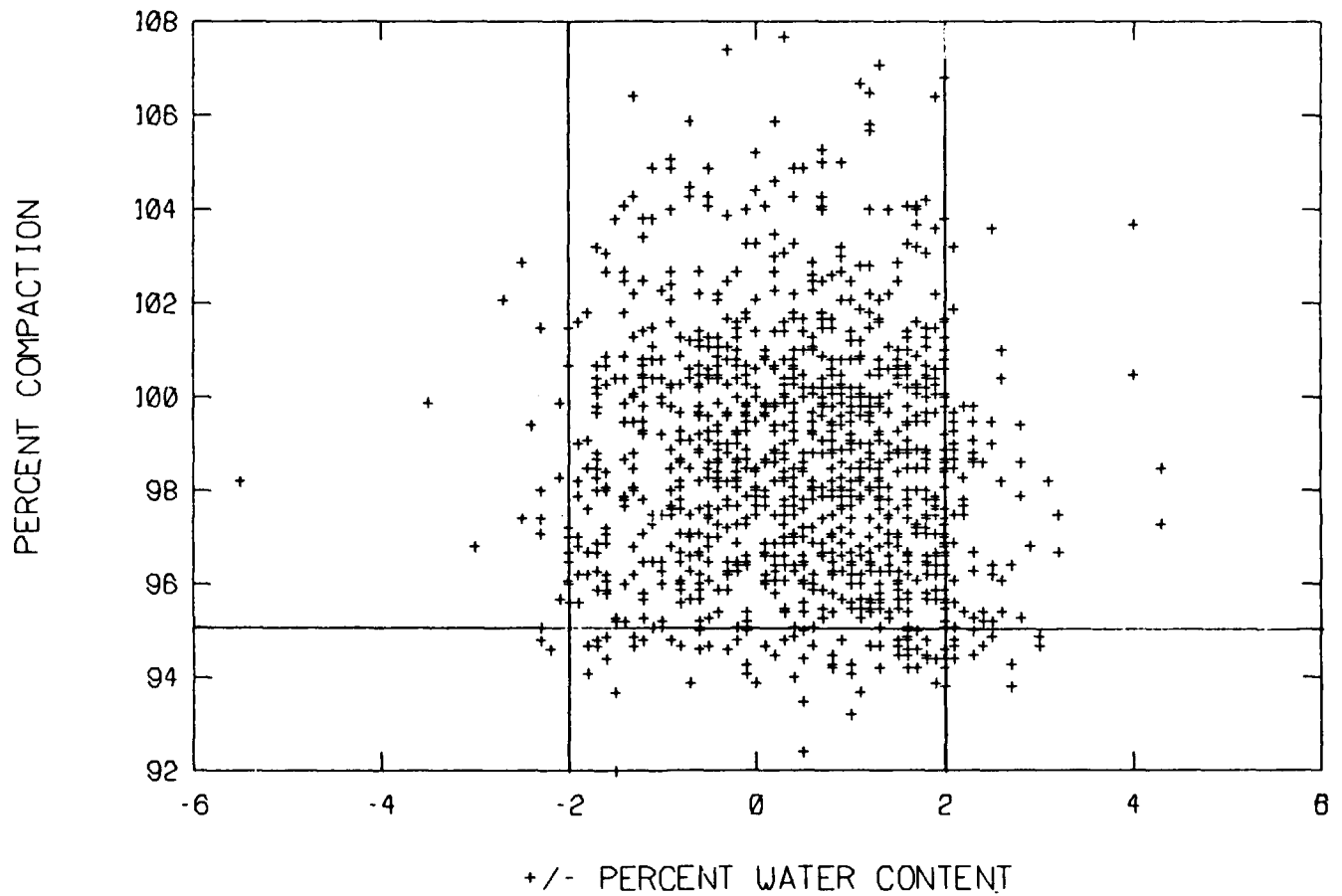
Figure 3. Scatter Plot of Percent Compaction and
Percent Water Content Compared to Proctor
Optimum for Compaction Control Data.
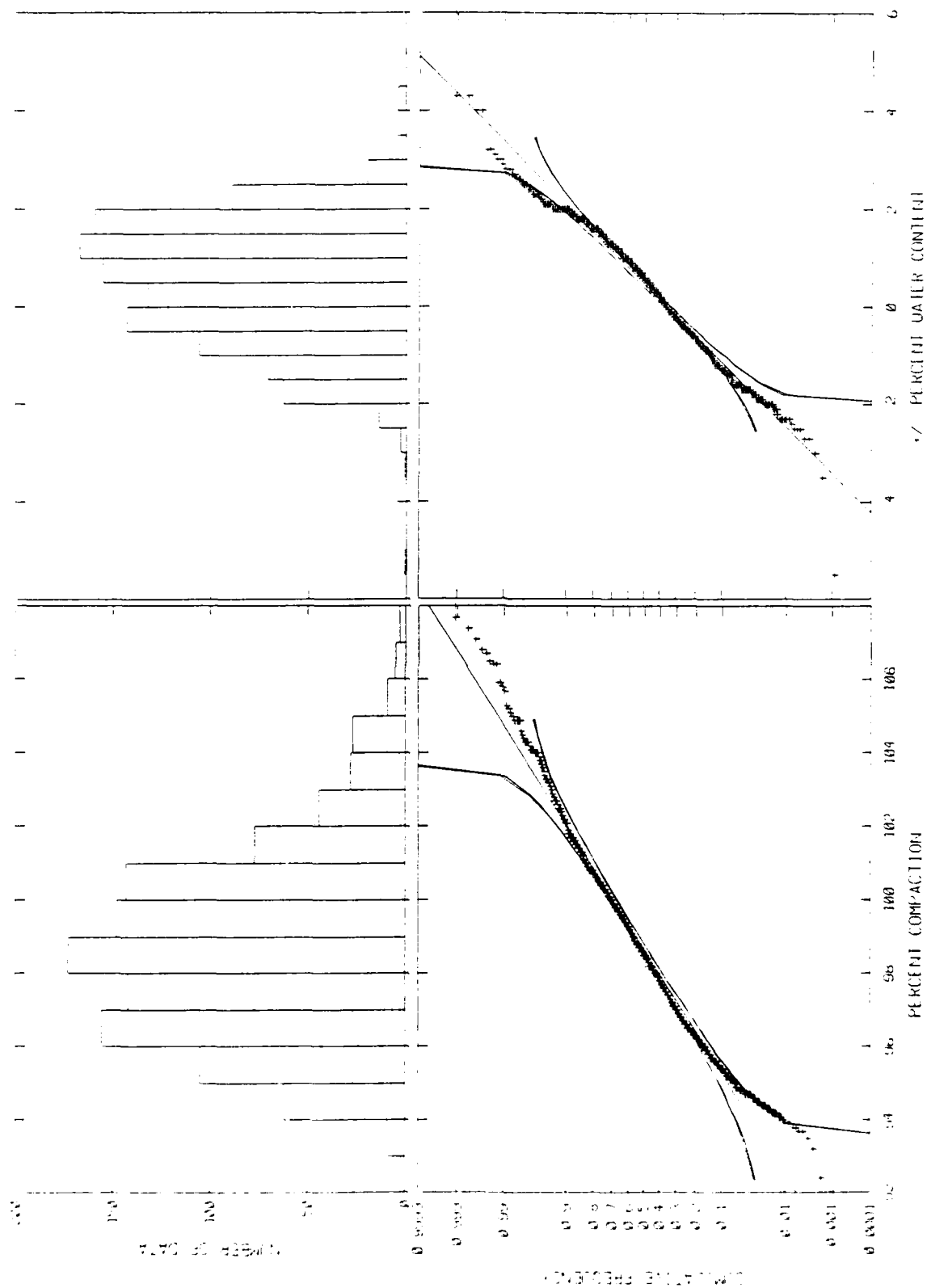
Figure 4. Histograms and Probability Paper Plots of the Compaction Control Data of Figure 3.
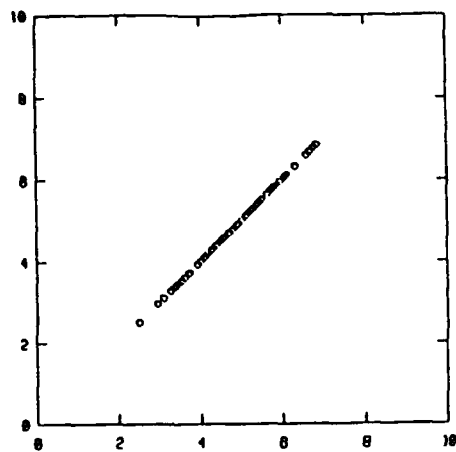
corr. coeff. · 1.0
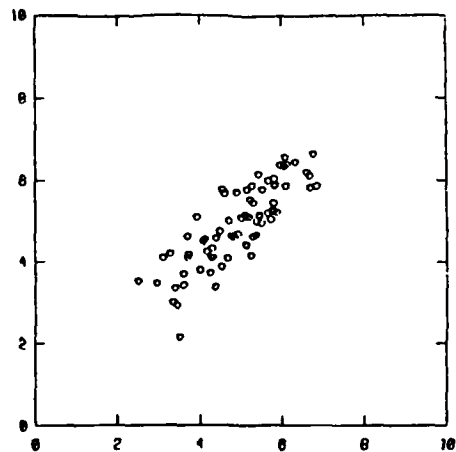
corr. coeff. · 0.8
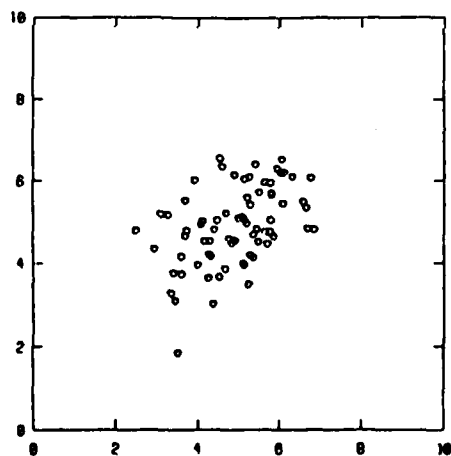
corr. coeff. · 0.4
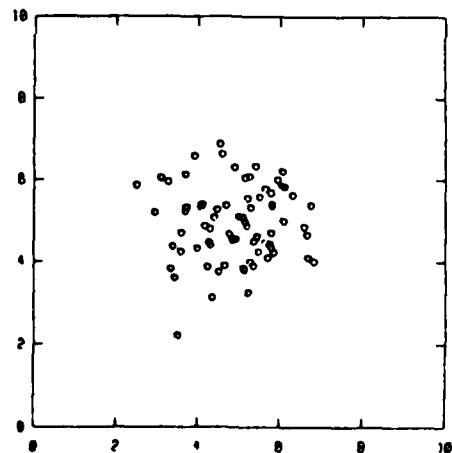
corr. coeff. · 0.0

Figure 5. Scatter Plots Showing Various Correlation Coefficients.
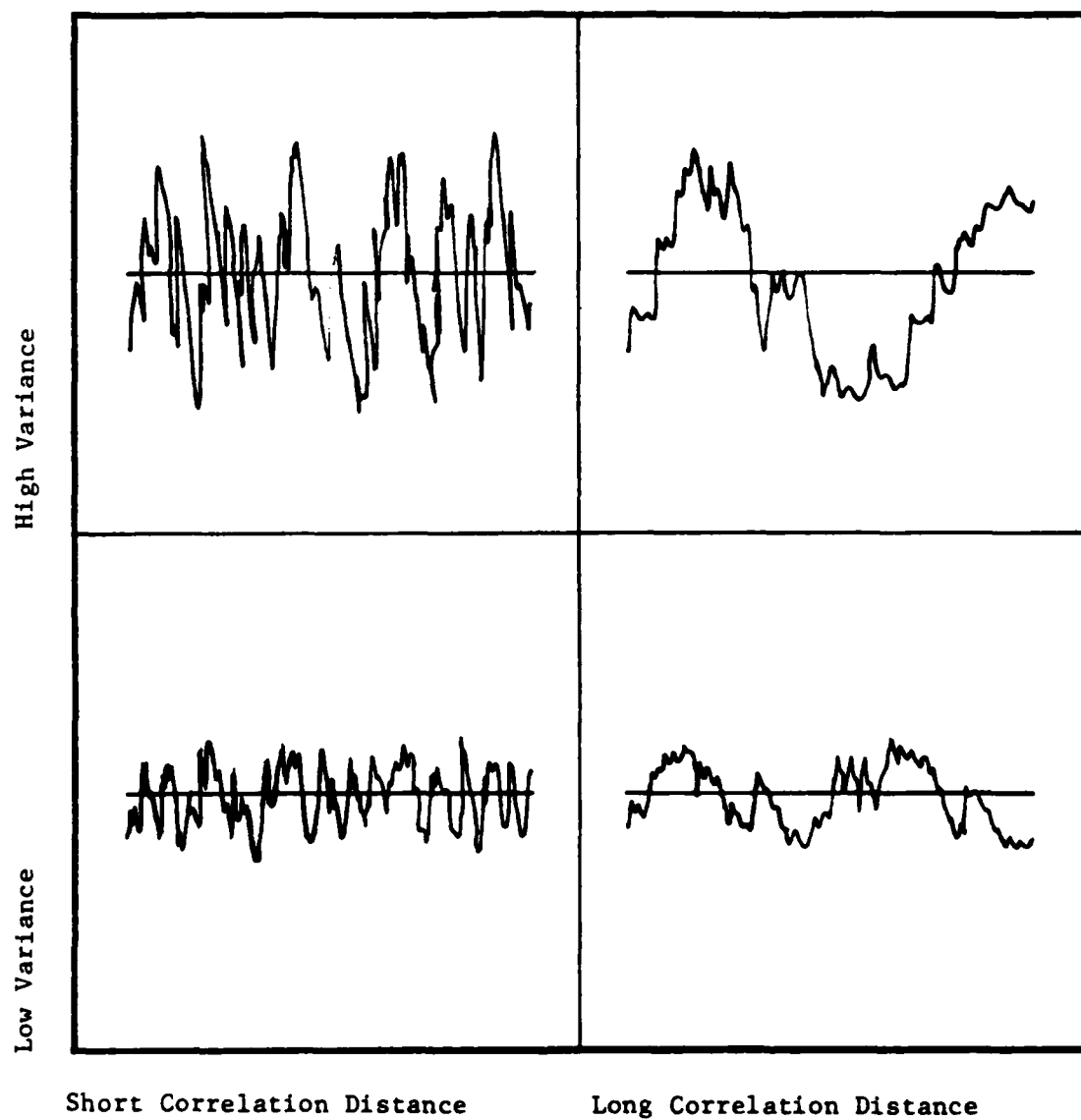
Figure 6.   Schematic Illustration of Variance and Autocorrelation in Soil Property Data.
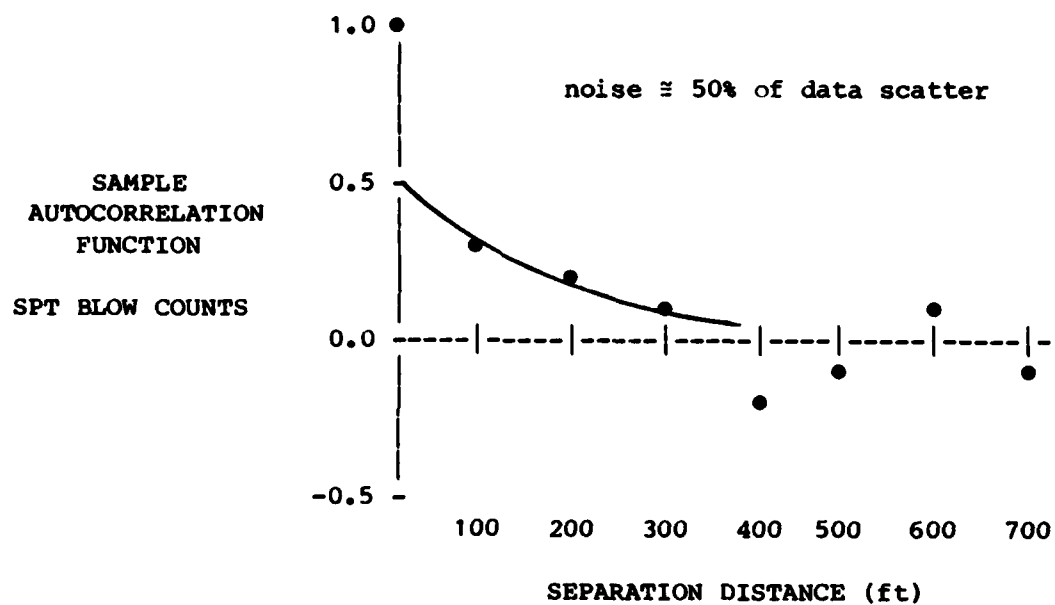
Figure 7. Autocorrelation Function for the Data of Figure 3.
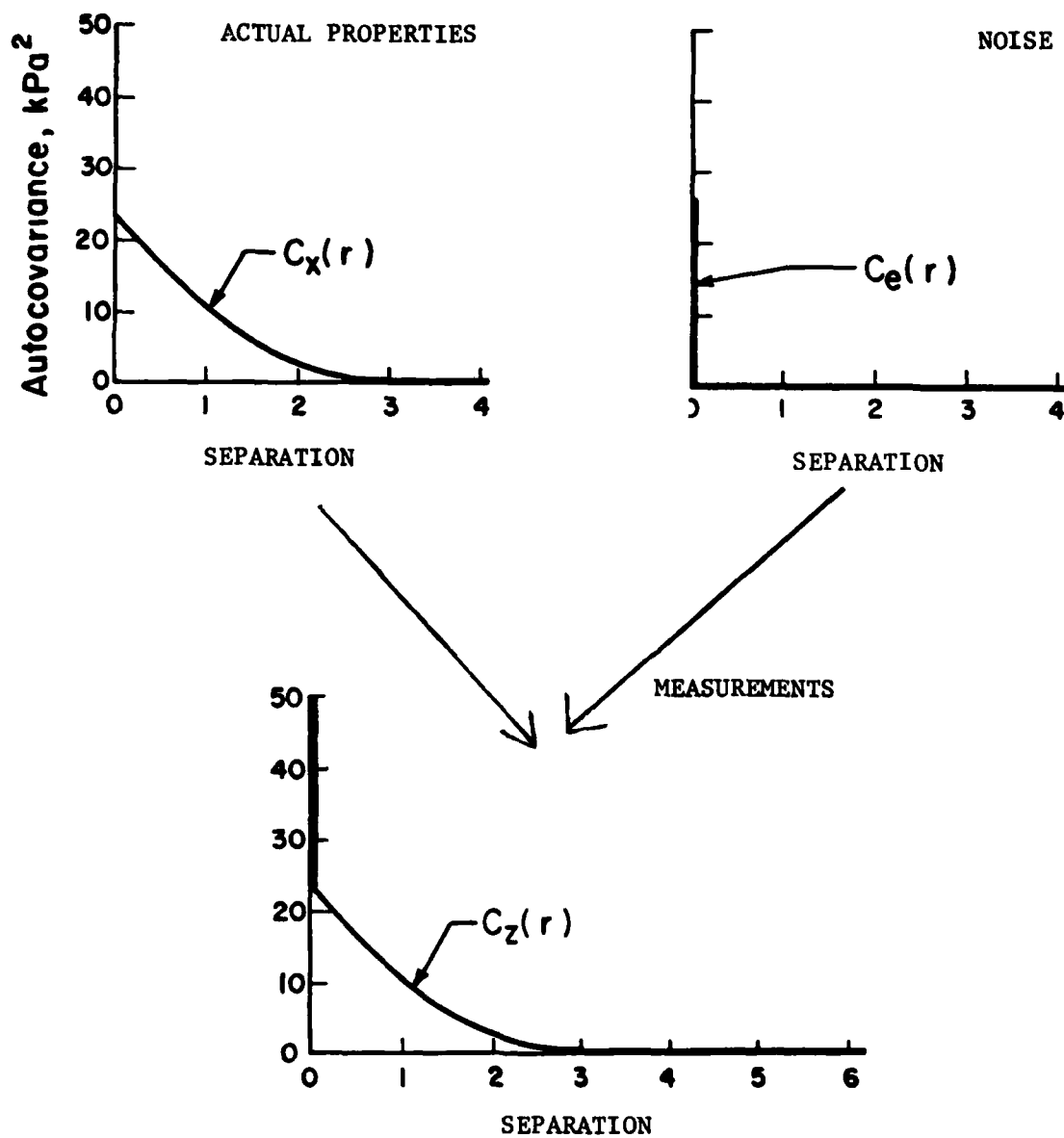
Figure 8. Estimate of Measurement Noise.

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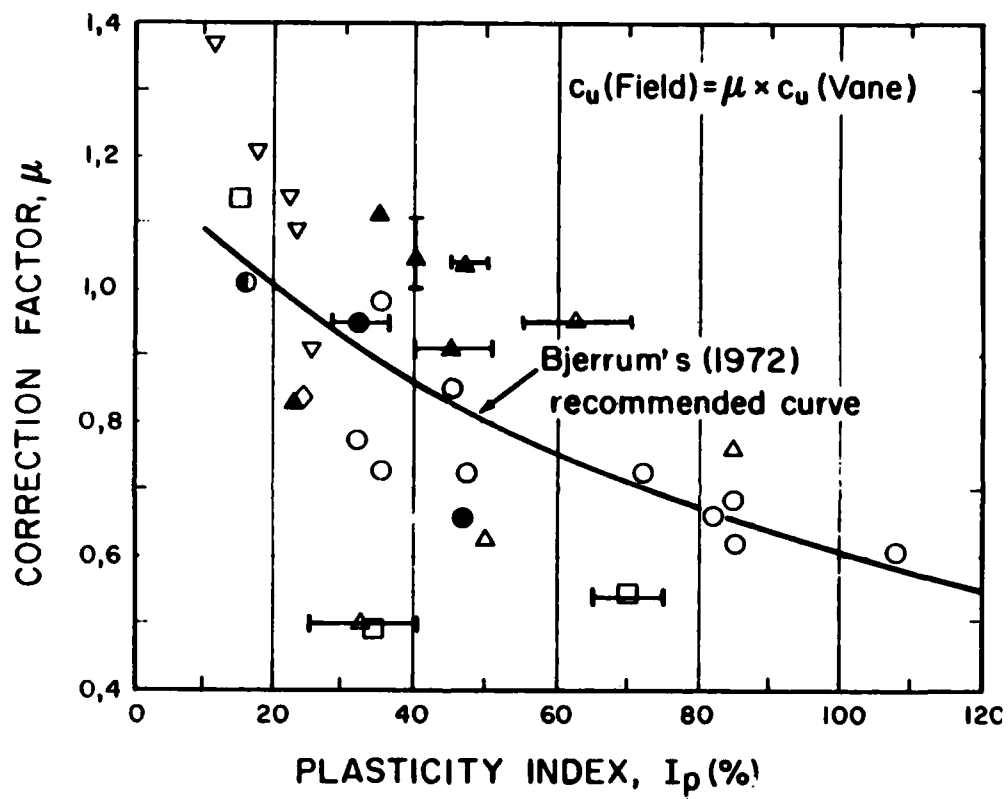
Figure 9. Bjerrum's Bias Error Correction Factor for Field Vane
Strength Measurements.
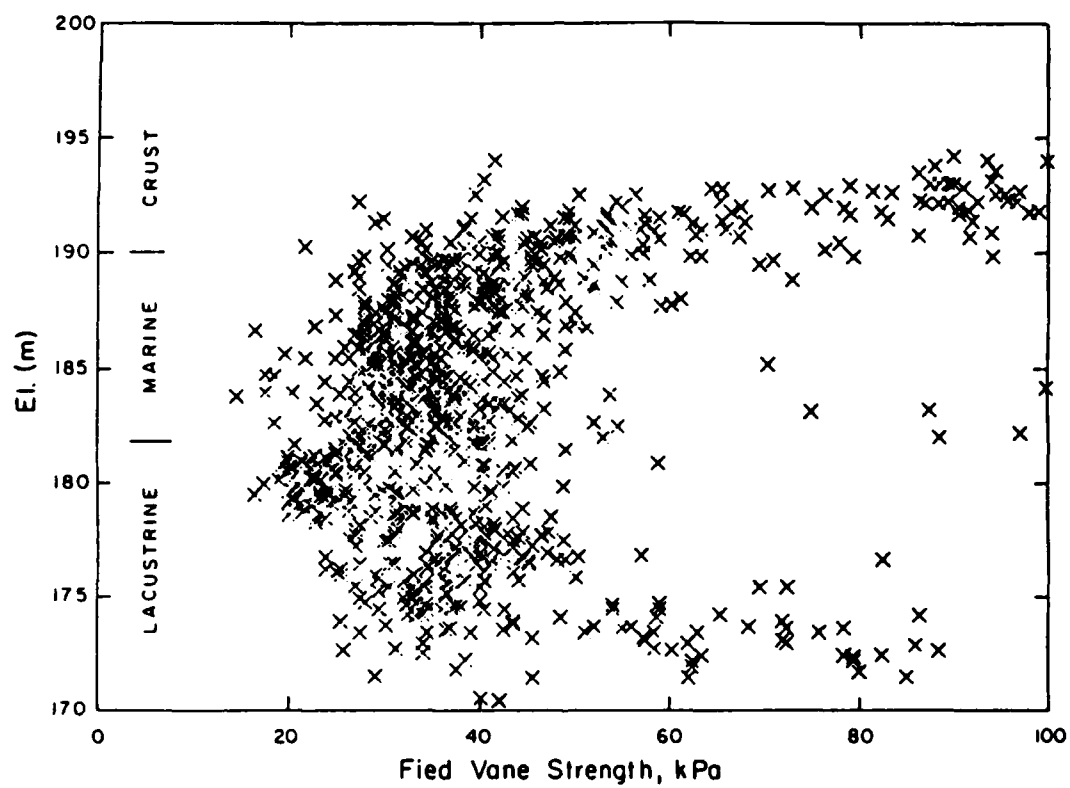
Figure 10. Field Vane Strength Data From A Soft Clay Deposit.

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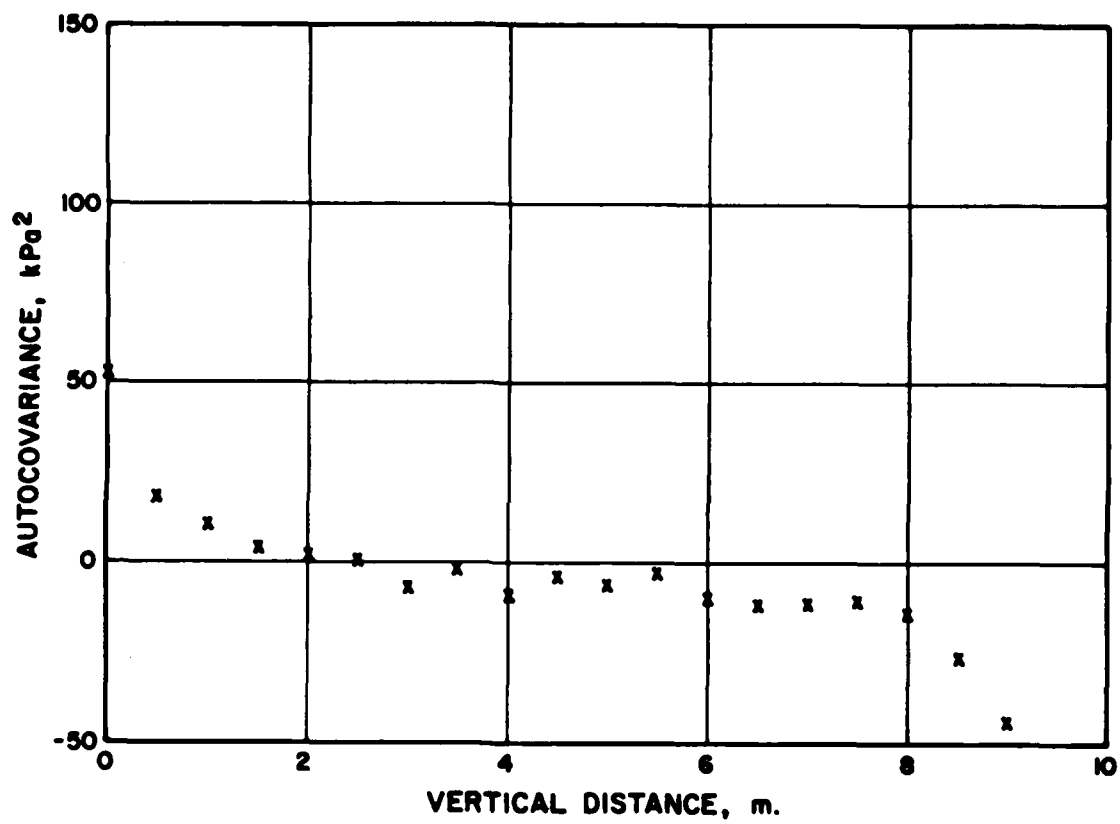
Figure 11. Vertical Autocovariance Function for the FV Data of Figure 10.
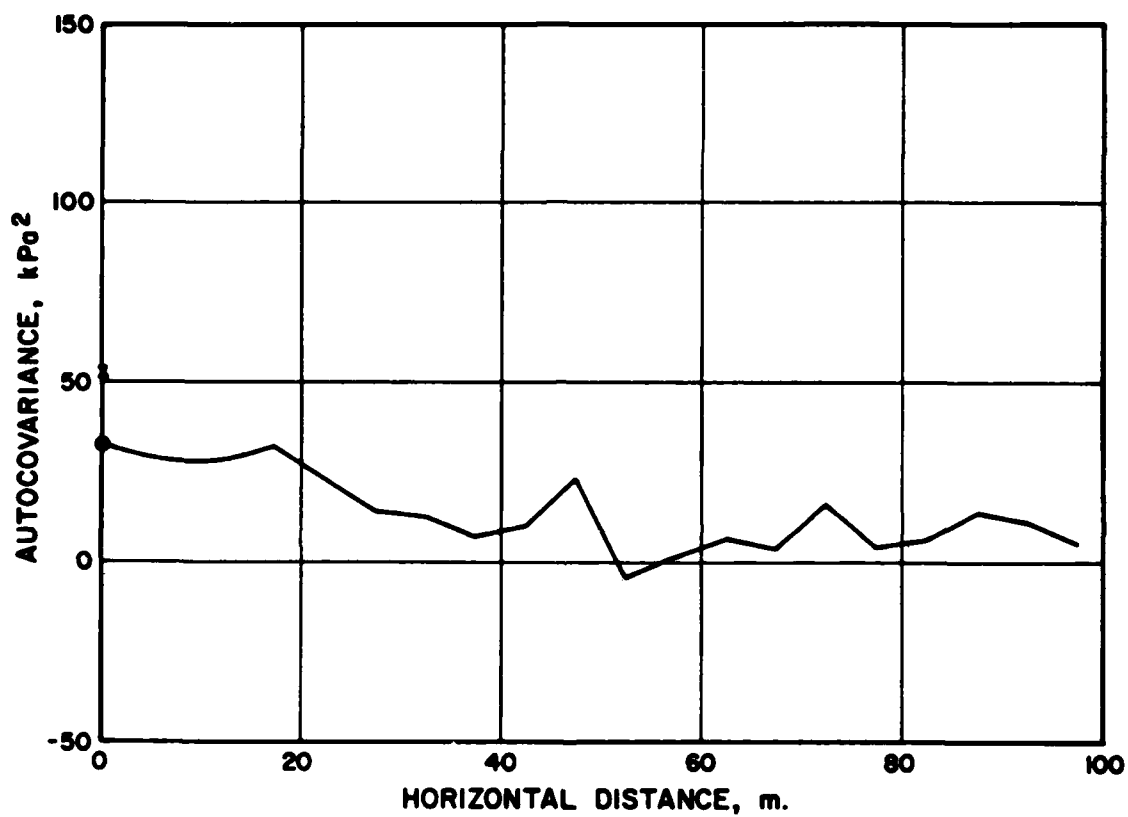
Figure 12. Horizontal Autocovariance Function for the FV Data o f
Figure 10.
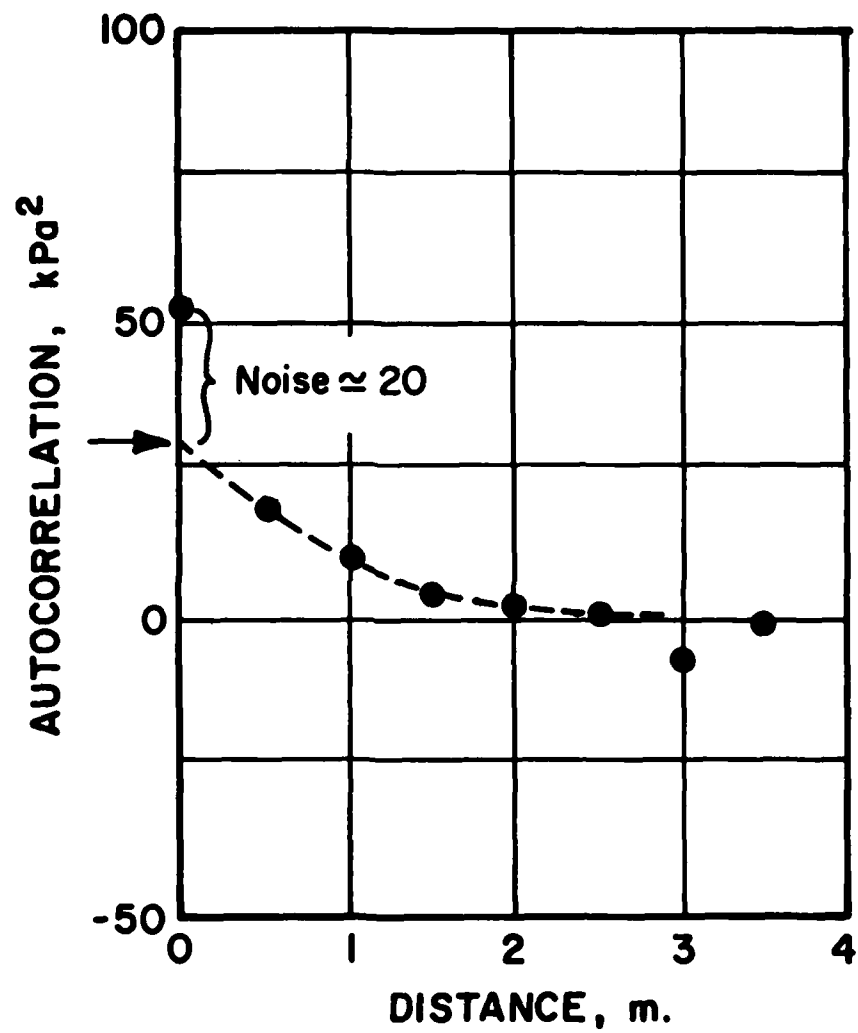
Figure 13. Estimate of Measure Noise in FV Data of Figure 10.

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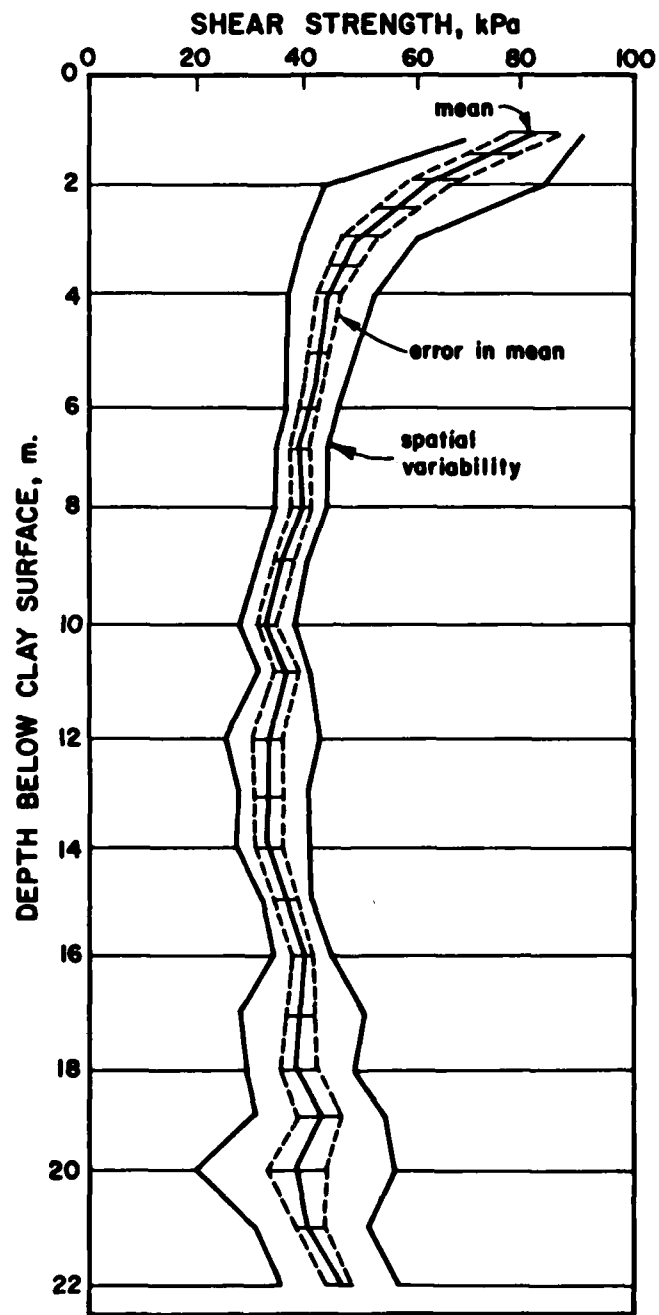
Figure 14. Statistical Design Profile from Data of Figure 10.
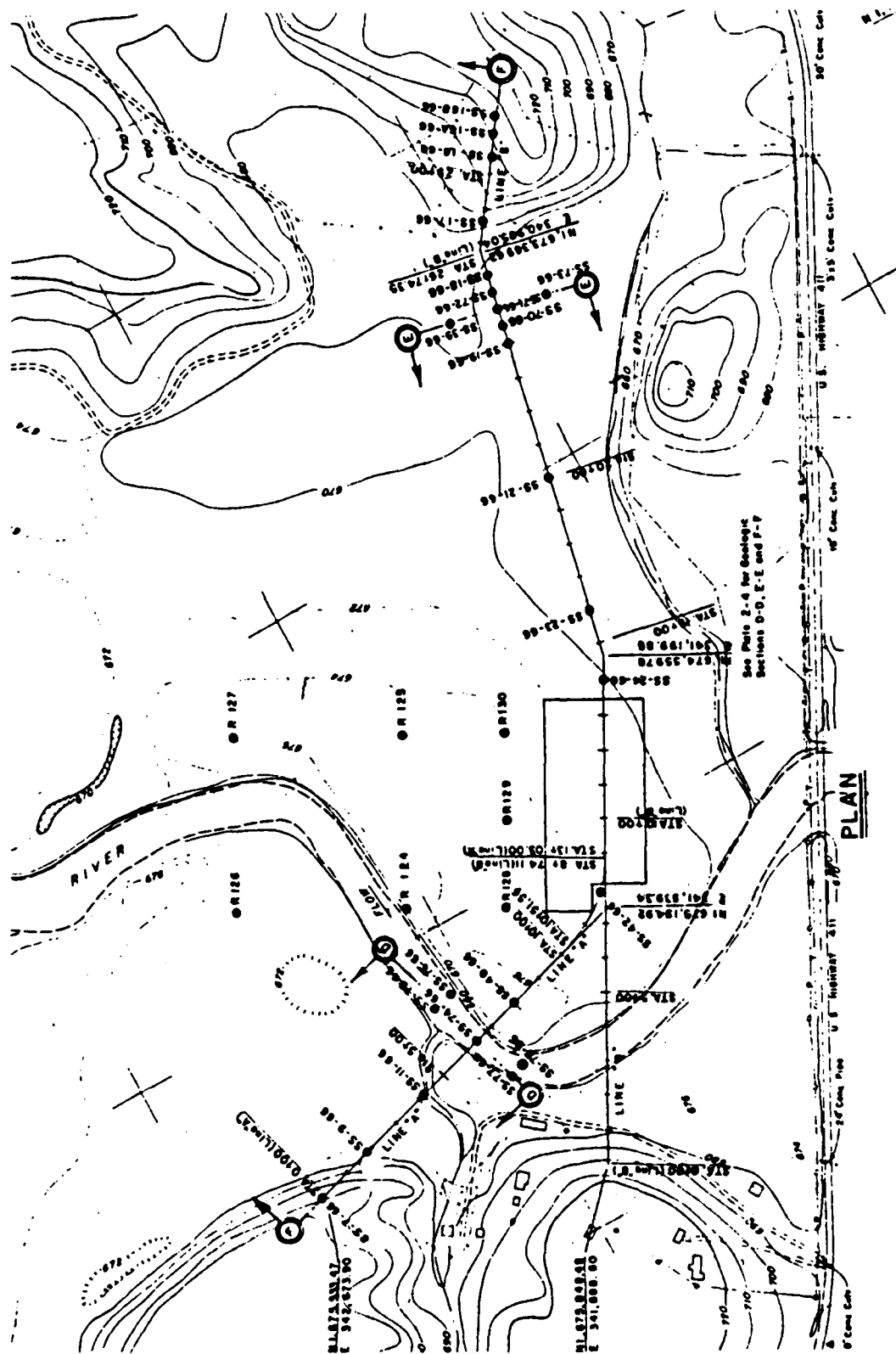
Figure 15. Boring Locations for SPT Data Collected at the Site of a Low Embankment Dam.

Figure 16. Interpretive Profile.

TOTAL NUMBER OF DATA · 188
MEAN · 4.8457   STD DEV · 2.8851
NUMBER OF OUTLIERS BEYOND LOWER LIMIT OF PLOT :   0
NUMBER OF OUTLIERS BEYOND UPPER LIMIT OF PLOT :   0

RE-REG DAM STA 4+00 - STA 13+00
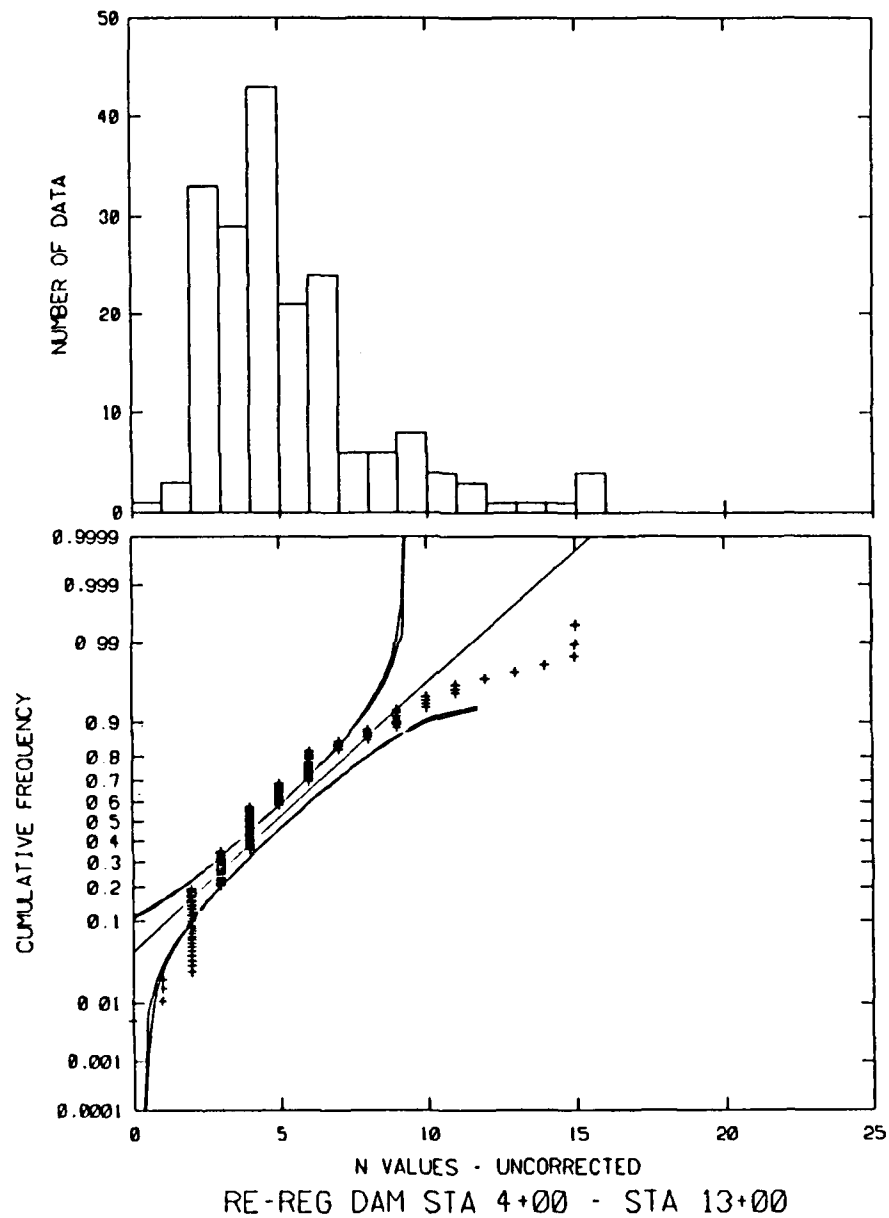
Figure 17.   Blow Count Data for Station 4+00 to 13+00.
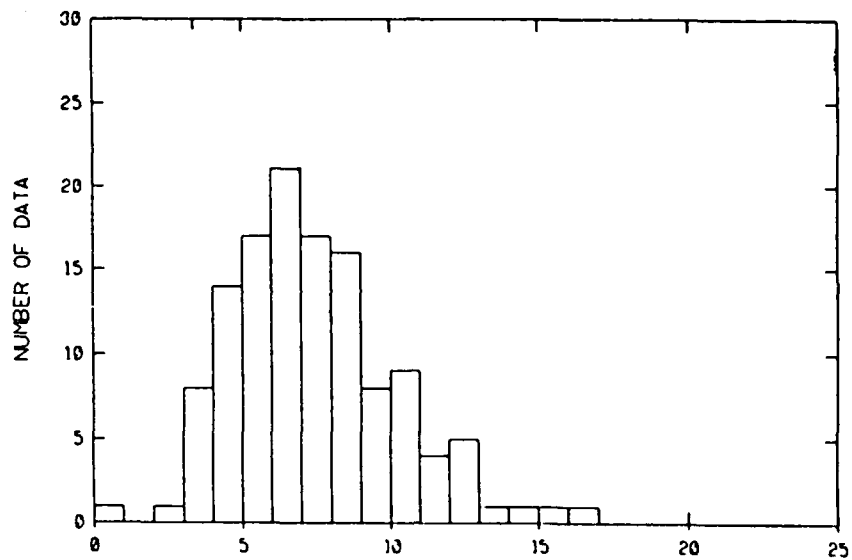
50
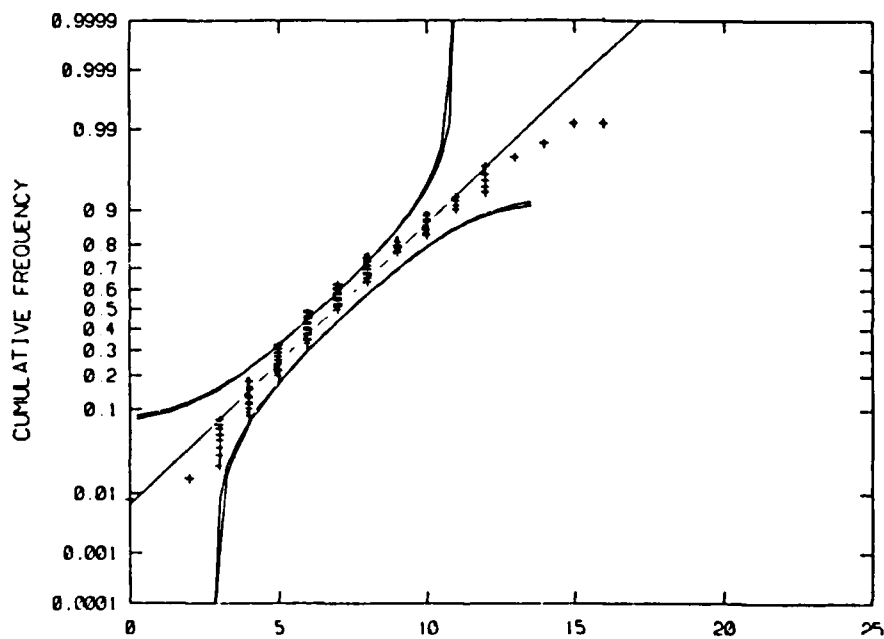
Figure 18. Blow Count Data for Station 17+00 to 24+50.
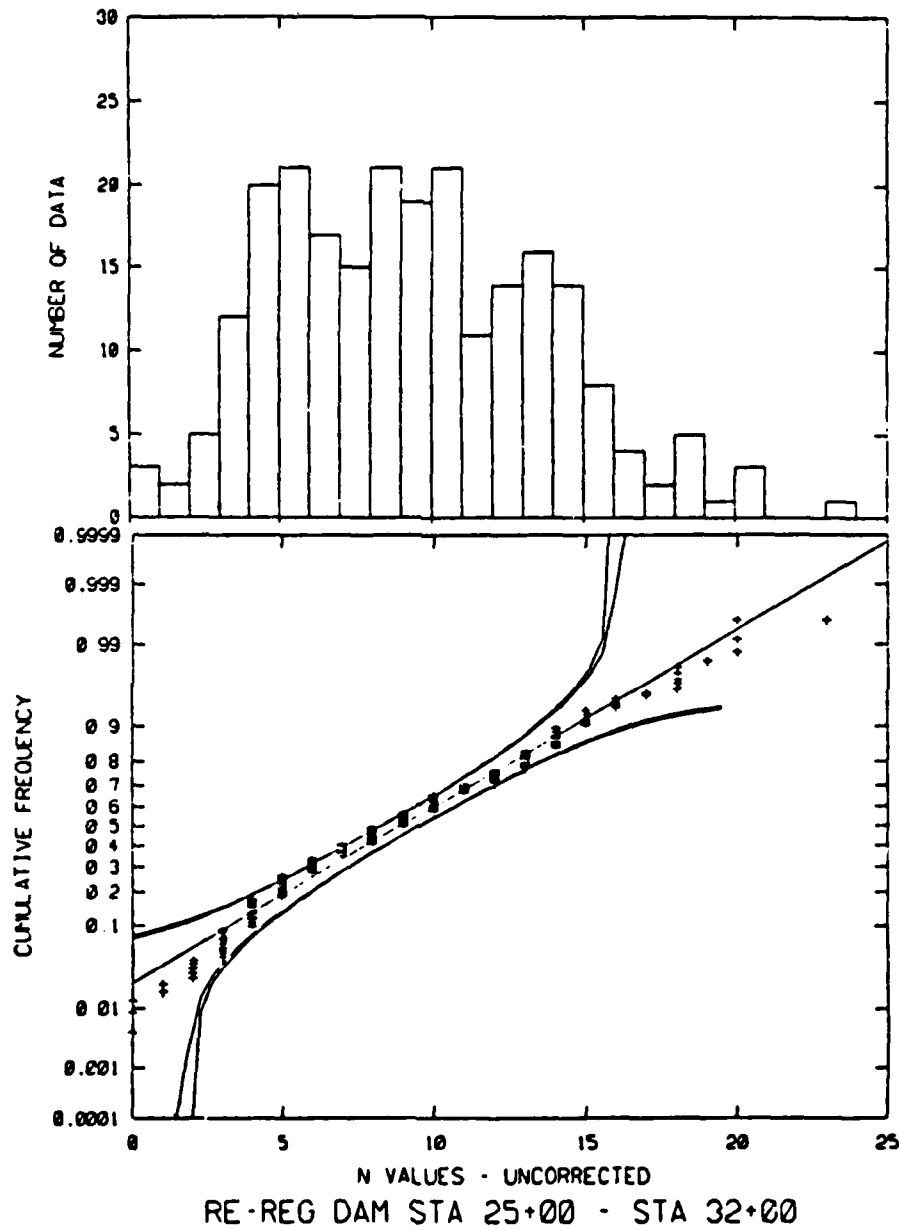
Figure 19. Blow Count Data for Station 25+00 to 32+00.
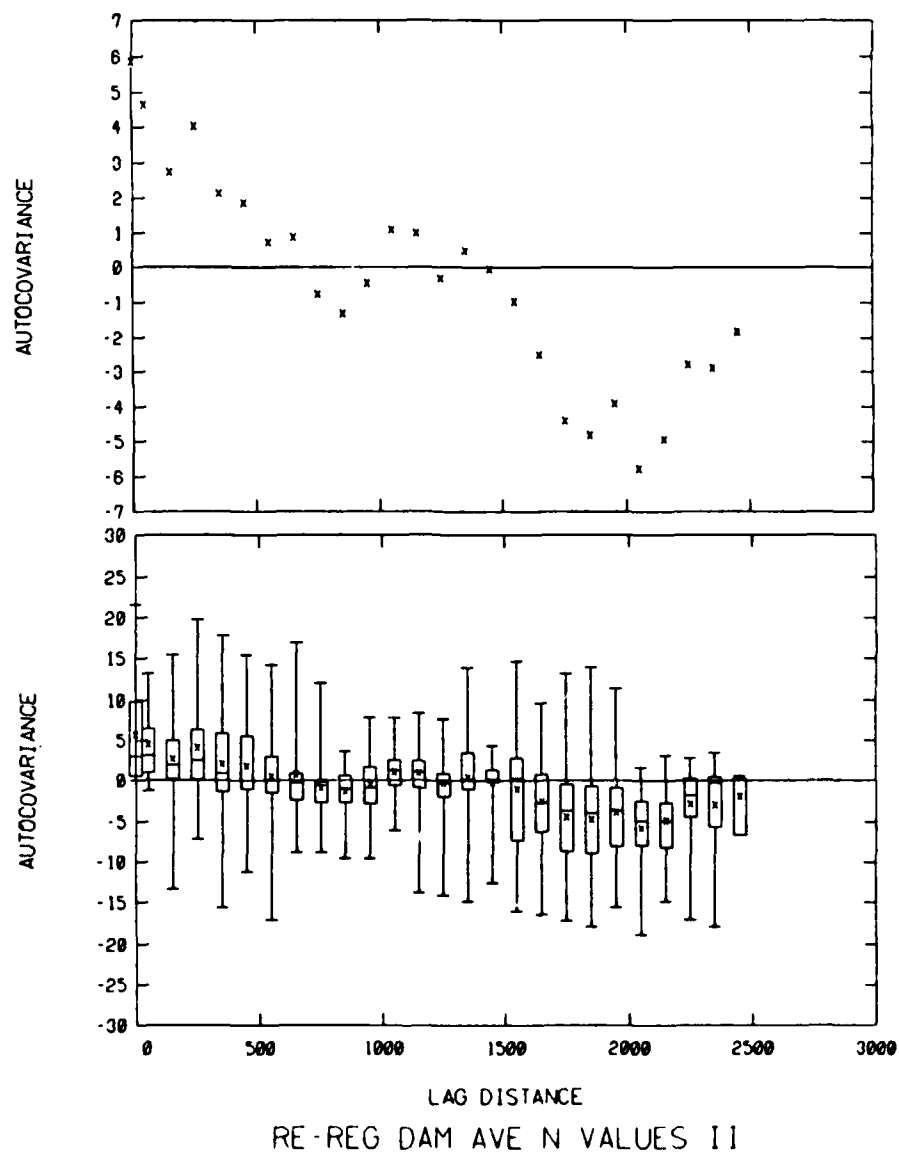
Figure 20. Horizontal Autocovariance Function for
Nondetrended SPT data.
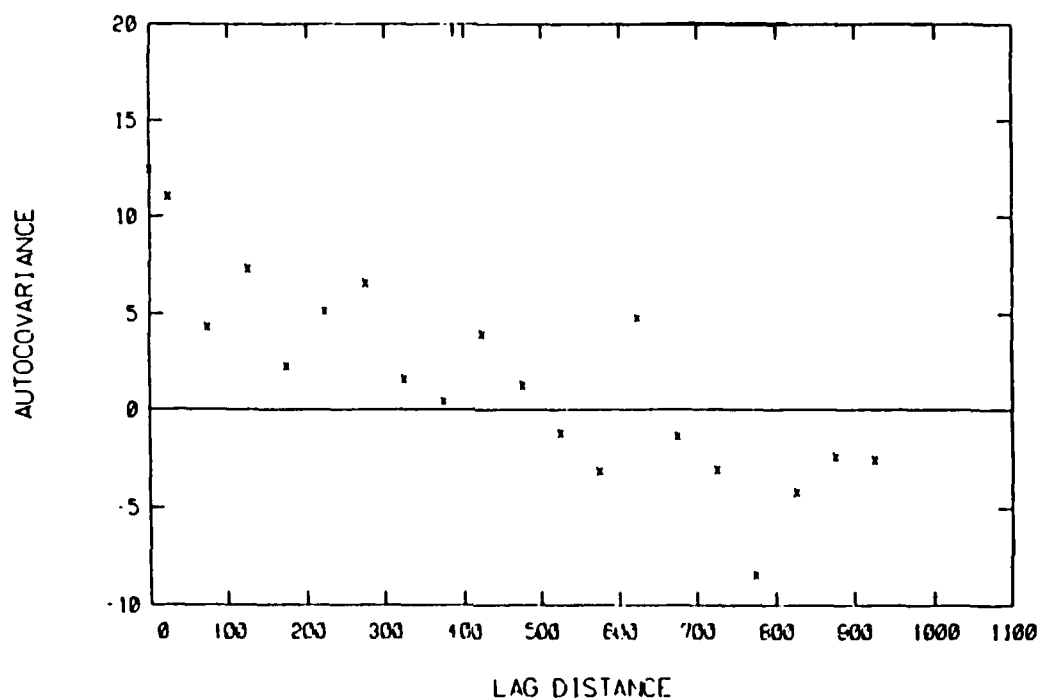
Figure 21. Horizontal Autocovariance for Individual SPT Data at
Elevation 660'.

STA. 4+ØØ - STA. 13+ØØ
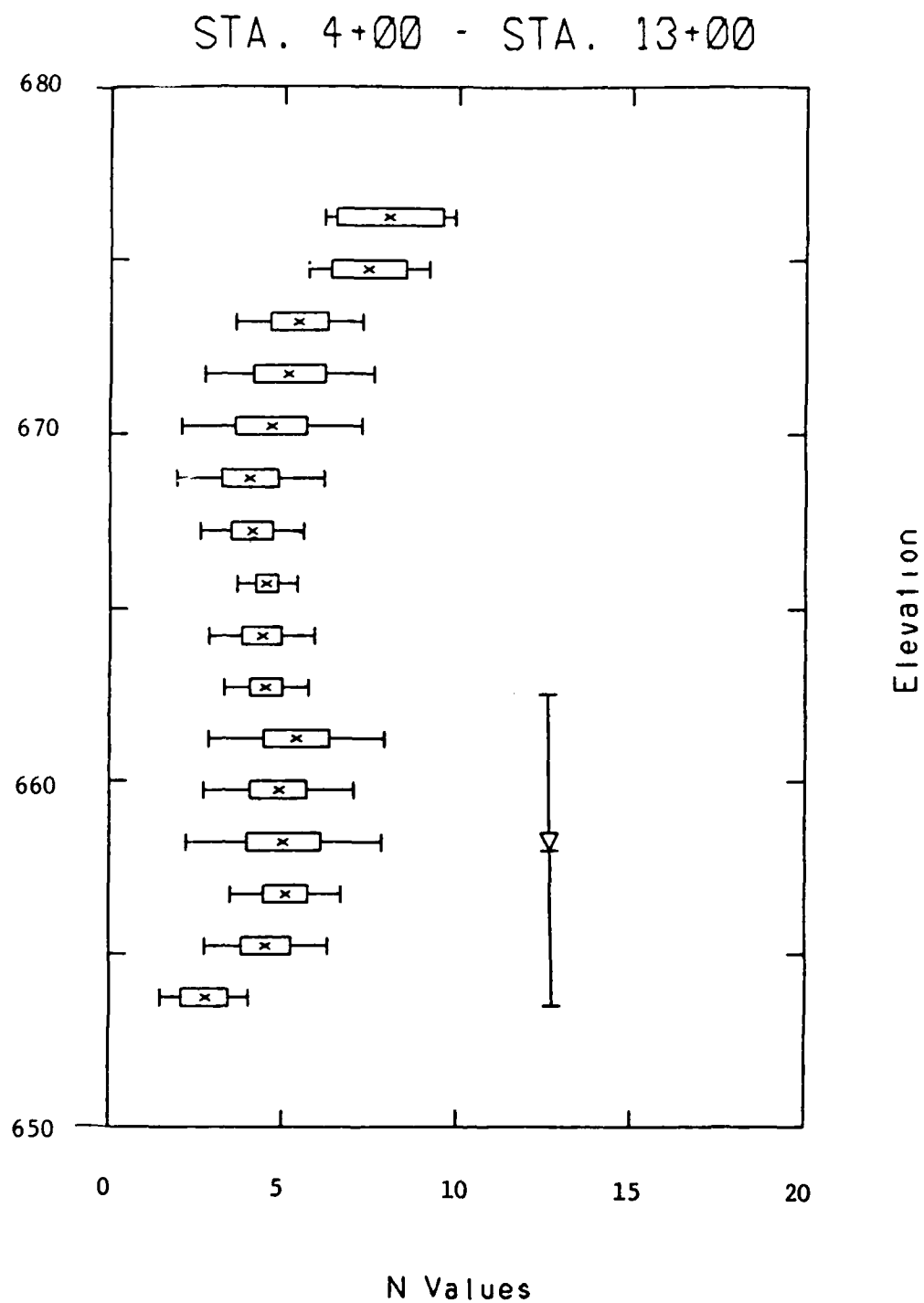
Figure 22.   Statistical Design Profile for Station 4+00 to 13+00.

STA. 17+00 - STA. 24+50
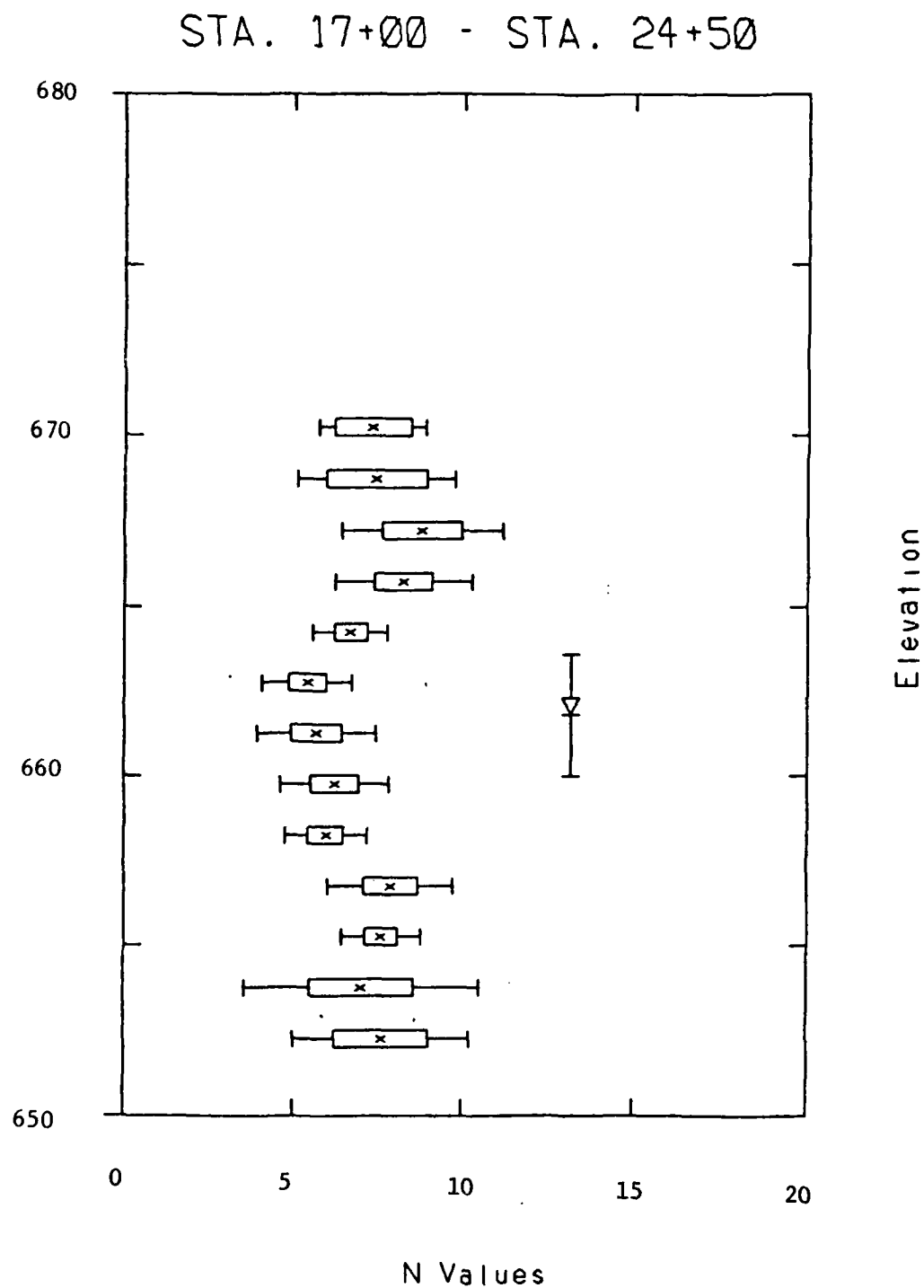
Figure 23.   Statistical Design Profile for Station 17+00 to 24+50.

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STA. 25+00 - STA. 32+00
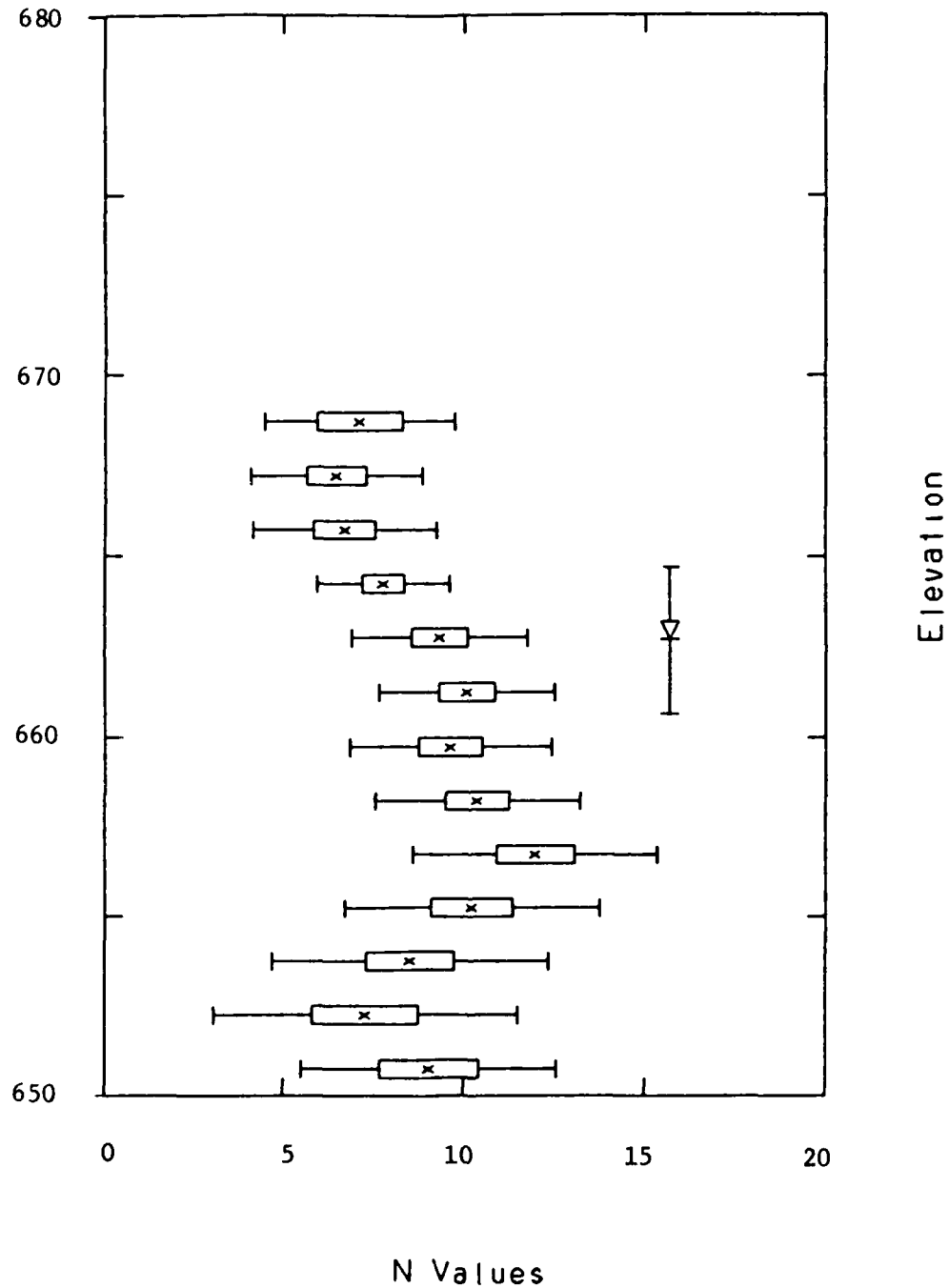
Figure 24.   Statistical Design Profile for Station 25+00 to 32+00.

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PART III: ERROR PROPAGATION IN ENGINEERING CALCULATIONS

This section explains the concept of error propogation in engineering calcuations and presents mathematical techniques for analyzing the magnitude of uncertainty in predictions.

A number of mathematical methods for error analysis are described. The most widely used of these methods is the first-order technique. This technique is described in greater detail. Four other methods are described briefly. These are point estimate techniques, response surface techniques, ajoint sensitivity, and Monte Carlo simulation.

Concept of Error Propagation Analysis

Engineering analysis uses soil property estimates made from measurements by incorporating them in models. These models are based on engineering mechanics and relate soil properties, loads, and other aspects of a design to predicted performance. Traditionally, point estimates of properties, loads, and other conditions are entered into the model and point estimates of performance are calculated. For example, to predict settlement of a footing on sand, data are used to make a best estimate of soil properties. This best estimate, perhaps modified to be conservative, is used as input to a settlement formula (i.e., a model). A best estimate of settlement is calculated from the formula as a function of load.

If errors have been made in estimating soil properties, then the settlement predicted by the formula will also be in error. The error in input is said to propagate through the model to cause an error in the output (Fig. 25). Sensitivity analysis is generally used to assess the effect of input errors on output. In sensitivity analysis a number of calculations are made

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using various estimates of input soil properties. The variation in calculated settlement caused by the variation in input soil properties indicates the sensitivity of the prediction to potential error in the input soil properties.

Sensitivity analysis works well when one input parameter is involved and when spatial variability of soil properties is not important. When more than one input parameter is uncertain or when spatial variability exerts a significant influence, sensitivity analysis does not work well. In these more complex situations sensitivity analysis provides no mechanism for considering combinations of uncertainties. In such cases a systematic accounting of the way errors in input translate to errors in output is needed. This systematic accounting is error analysis.

With error analysis, all calculations are based on best estimates of input parameters, avoiding conservatism as much as possible. The output of a calculation is the corresponding best estimate of facility performance. Uncertainty is incorporated using standard deviations and correlation coefficients. The techniques of error analysis allow the effect of standard deviations and correlation coefficients on input parameters to be translated to corresponding standard deviations and correlation coefficients on output (i.e., performance predictions). These standard deviations and correlation coefficients express the uncertainty or potential error in a calculated prediction. Using error analysis the joint result of a settlement calculation is a best estimate of settlement and a standard deviation on settlement.

First-Order Technique

The most common approach to error propogation is the first-order technique. This is also sometimes called the first-order second-moment method

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(FOSM). This technique is based on a linear approximation to the model which relates soil properties to performance predictions, and hence is 'first-order'. Because the technique uses only the means, standard deviations, and correlation coefficients of the soil properties, it is said to be a 'second-moment' method.

Consider the model relating an input soil parameter x to the performance prediction y through some form of equation(s),

$$y = g(x) \tag{26}$$

The model $g(x)$ can be analytical, numerical, empirical, etc. This model can be linearized by expanding the right hand side (RHS) in a Taylor's series about some point $x=\eta$,

$$y = g(\eta) + \frac{\dot{g}(\eta)(x-\eta)}{1!} + \frac{\ddot{g}(\eta)(x-\eta)^2}{2!} + \dots \tag{27}$$

If the RHS is truncated to the first two terms, an approximation of y as a linear function of x is obtained.

Mean or Best Estimate Prediction

Applying probability theory to the truncated version of Eqn. 27, the following result is obtained:

$$m_y \doteq g(m_x) \tag{28}$$

in which $\doteq$ indicates first-order (i.e., linear) approximation. In words, the mean or best estimate of the prediction y is the function of the mean or best estimate of the parameter x. This is the normal deterministic solution using

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the best-estimate (mean) soil property as input. If the prediction of y

depends on a set of parameters $\{x_1, \ldots, x_k\}$ the equivalent form of Eqn. 28 is,

$$m_y \doteq g(m_{x_1}, \ldots, m_{x_k}) \tag{29}$$

in which m_{x_1} = the mean of x_1, and so on.

Uncertainty (Standard Deviation) in Predictions

Again applying probability theory to Eqn. 27, a second result is obtained

concerning the relationship of the standard deviation of the prediction y to

the standard deviation of the input parameter x,

$$s_y \doteq (\frac{dy}{dx}) \; s_x \tag{30a}$$

$$v_y \doteq (\frac{dy}{dx})^2 \; v_x \tag{30b}$$

In words, the standard deviation of the prediction y is the product of the

standard deviation of the parameter x and the derivative of y with respect to

x. The derivative of y with respect to x might be thought of as the

sensitivity of y to changes in x. By squaring both sides a relationship is

obtained between the variance of y and the variance of x. Again, these results

are based on a linear approximation to $g(x)$, but for most geotechnical problems

they are sufficiently accurate.

Plate 1 shows a simple calculation of bearing capacity for an unembedded

footing in which the only source of uncertainty considered is data scatter.

The data on friction angle are taken from laboratory tests, and bearing capacity is calculated using Terzaghi's bearing capacity factor N_γ. For illustration, this factor is related empirically to ϕ' by the approximate equation $N_\gamma \cong 0.01e^{0.25\phi'}$ (Scott, 1963). The mean value of bearing capacity is found by substituting the mean value of $\tan \phi'$ in the empirical equation and then into the bearing capacity formula. A more complete analysis replacing Scott's formula by experimental data is given by Ingra and Baecher (1983).

If the prediction y depends on a set of parameters, the equivalent form of Eqn. 30 is,

$$V_y \doteq \Sigma \; \Sigma \; \frac{dg}{dx_i} \; \frac{dg}{dx_j} \; C_{x_i, x_j} \qquad (31)$$

in which C_{x_i, x_j} = the covariance of x_i and x_j. When all the x_i and x_j are independent of one another, each of the covariance terms for $i \neq j$ is zero. The covariances for $i=j$ by definition are simply the variances of the x_i (Cf., Eqns. 2 and 6). Thus, for this special but common case Eqn. 31 reduces to,

$$V_y \doteq \Sigma \; (\frac{dg}{dx_i})^2 \; V_{x_i} \qquad (32)$$

Plate 2 shows a slope stability calculation for an embankment constructed on soft clay. The field vane data for these foundation soils are shown in Fig. 10.

Two other special cases deserve note because they are common in practice and lead to simple results. For the case in which y is a linear combination of a set of independent parameters $y = \Sigma a_i x_i$, Equation 32 becomes,

$$V_y = \Sigma\, a^2\, V_{x_i} \tag{33}$$

For the case in which y is a power function of a set of independent parameters, $y = \pi\, x_i{}^{a_i}$, Equation 32 becomes, approximately,

$$1 + \Omega_y{}^2 \doteq \Sigma(1 + a_i{}^2 \Omega_{\dot{x}}{}^2) \tag{34}$$

which for small coefficients of variation ($\Omega_x < 0.3$) reduces to,

$$\Omega^2{}_y \doteq \Sigma a_i{}^2\, \Omega^2{}_{xi} \tag{35}$$

Correlations Among Predictions[†]

The general form of Eqs. 20 and 21 when a set of predictions $\underline{Y} = \{y_1, \ldots, y_h\}$ is calculated from a set of soil properties $\underline{X} = \{x_1, \ldots, x_k\}$ is,

$$m_Y = g(m_X), \text{ and}$$

$$\Sigma_y = G^t \Sigma_x G,$$

in which $my = \{m_{y1}, \ldots, m_{yh}\}$, Σ_Y = the covariance matrix of $\underline{Y}$, and Σ_X is the covariance matrix of $\underline{X}$. Σ_Y has ij^{th} term $C_{yi,yj}$ and Σ_X has ij^{th} term $C_{xi,xj}$. G is the matrix of derivatives with ij^{th} term dy_i/dx_j. The diagonal terms of Σ_Y give the variances of $y_1, \ldots, y_h$, the off-diagonal terms give the covariances of y_i, y_j. The correlation between y_i and y_j can be found from Eqn. 7.

[†] This section is given for completeness, but may be skipped on first reading. A more complete presentation of advanced topics in error analysis is given by Ditlevsen (1981).

Size Effect Factor

The volume of soil influenced by an in situ test or contained in a laboratory specimen is small compared with that influenced by a prototype structure. To make predictions of how the prototype will perform, one needs to estimate average properties within this larger representative volume of soil, and the variability among the average properties of representative sized volumes.

This is done by assuming the representative volume to be composed of a large number of smaller elements, for example, each the size of a test specimen. From the formulas in Part II the mean and standard deviation of the properties of specimen sized elements are calculated, then using the spatial structure described by the autocorrelation function, a mean and standard deviation for the larger representative volumes is calculated. These calculations are summarized in a size-effect factor, R, which in many cases can be expressed by simple formulas or can be tabulated.

Spatial Averaging

The most important application of the size effect factor occurs in the case where average properties within a large volume of soil control the engineering performance of a facility.

Empirically, the variability of the average soil properties within small elements of soil is larger than the variability of the average properties within large elements. Within a small volume physical properties tend to be more or less uniform throughout. Some individual elements may have greater than average strength, say, while some may have less than average, but within

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any one element there is little variability. There is more variability among the average properties of different elements than there is within a single element. Within large volumes the opposite is true, there tends to be a mixture of high and low properties. Thus, with small volumes the properties of individual elements may vary sharply from the mean across the site, but with large volumes internal variations balance out and the average property from one large element to another differs very little. The mean of large volumes remains the same as the mean of small volumes, but the standard deviation of the average property from one large volume element to the next is small. There is more internal variability within large elements than there is among the average of one element to the next.

The extent of averaging of properties within a large volume of soil depends on the structure of the spatial variation of the soil properties. More precisely, the extent of averaging depends on the standard deviation of properties from point to point and on the autocorrelation function.

Consider the one-dimensional problem of calculating the variability of average SPT blow count among borings in a homogeneous soil. Plate 3 shows a set of six boring logs. One N value is randomly chosen from each boring and the standard deviation among them is calculated. Then two N values in each boring are randomly chosen, the average of the two for each boring is calculated, and the standard deviation of the boring averages taken. This calculation with two values from each boring gives a somewhat smaller standard deviation than the calculation with only one N value. Continuing in the same way, the greater the number of N values included in the average for each boring, the smaller the standard deviation of the boring-averaged N across the six borings.

From Eqn. 12 one should be able to predict this decrease in standard deviation as the number of terms in each boring average is increased. Namely, the standard deviation of the boring averages ought to decrease by $1/\sqrt{k}$ as the number of N values in each boring, k, increases. This assumes that the blow counts are mutually independent (i.e., their correlation coefficients are zero). If the blow counts are (auto)correlated, then the standard deviation will still decrease, although not as quickly as $1/\sqrt{k}$. The decrease of the standard deviation of average blow count as the number of N values included in each average increases is a manifestation of spatial averaging. The larger the volume of soil (i.e., the greater the number of values in each average) the more the individual fluctuations balance out.

The size effect factor, R, for the averaging case is defined as the ratio of the variance of the average soil property within a large volume of soil to the variance among test-sized volumes,

$$R = V_{m_X}/V_N \tag{36}$$

in which Vm_X is the variance of the mean of the k blow count values, $m_N=(1/k)\Sigma N_i$. The ratio of variances rather than standard deviations is used because it is more convenient for subsequent calculations. When data are autocorrelated the size effect factor R decreases more slowly than $1/\sqrt{k}$ in the example above, because the data are somewhat correlated. That is, the data show 'wavy' variations about the spatial mean and therefore the balancing out of spatial variations takes place more slowly.

Knowing the autocorrelation function, the exact shape of the relation of R to k can be calculated. Let the average blow count within a boring be

calculated as,

$$m_{Nj} = \frac{1}{k} \sum_{i=1}^{k} N_{ij} \tag{37}$$

in which N_{ij} is the ith blow count in boring j, and m_{Nj} =average of the blow counts in boring j. Presume that if a very large number of blow counts were measured within the same stratum, the overall mean and standard deviation would be m_N and s_N. Then, if the blow counts within each boring were widely spaced--and therefore independent of one another--the standard deviation of m_{Nj} across the n borings j=1,...,n would be

$$s_{mN} = s_N / \sqrt{k} \tag{38}$$

The average of the n_{Nj} over many borings would simply be m_N.

If more than k blow counts were measured in each boring, keeping the total length of each boring constant, fairly soon the assumption that the data are independent would break down. Indeed, were it physically possible to make measurements very close together, as the separation between measurements approached zero the correlation between the data would approach 1.0. Such correlation can be accounted for mathematically using a modified version of Eqn. 12 to derive the standard deviation of the mean of a set of data.[*] This is tedious for discrete data, but for the continuous case which is really of more interest the mathematical results are uncomplicated.

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[*] Specifically, for data $\underline{X} = (x_1,...,x_n)$ which exhibit correlations as reflected in a covaraince matrix $\underline{C}_X$, of which the ij^{th} term is (Cx_i,x_j), the variance of the average $m_x = (1/n)\Sigma x_i$ over different samples of n data is $Vm_x = (1/n)^t \underline{C}_X (1/n)$, in which $(1/n)$ = a vector of dimension n, each element of which is $1/n$ (Snedecor and Cochran, 1964).

The continuous case considers the variability of the average of soil properties which themselves vary continuously along a line or within a volume of soil. This, for example, is the situation faced in predicting the variability among footing settlements caused by variation in compression modulus in foundation soils, or in predicting the variability in factor of safety against instability along an embankment caused by variation in strength parameters of the fill.

Fig. 26 shows the effect of spatial averaging in one dimension as a function of the length over which the averaging takes place. The various curves apply, respectively, to patterns of spatial variability in soil properties which are represented by various analytical models for the autocovariance function. The horizontal axis shows the window length over which averaging takes place, normalized to the scale parameter of the autocovarance function. The vertical axis shows the size effect factor R. Similar graphs for 2D and 3D averaging, and procedures for obtaining R in special circumstances are given in the report, "Statistical Analysis of Geotechnical Data" (Contract Report GL-87-1).

A handy trick for continuum problems is to note that for many of the covariance functions used in practice the one-dimensional size-effect factor R asymptotically decays as,

$$R = \frac{V_{m_{xi}}}{V_{m_x}} + \frac{1}{L/2\delta_0} \tag{37a}$$

in which L is the depth over which the averaging takes place, and δ_0 is the autocorrelation distance. For L/δ_0 greater than about 2 this approximation is satisfactory.

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On the other hand, note that if the soil profile were divided into n layers and the layers were taken as mutually independent, then by Eqn. 12 the variance of the average m_X over all n layers would decay as,

$$R = 1/n \qquad\qquad (37b)$$

Therefore, if a continuous stratum were replaced by independent layers chosen to have thickness $L/2\delta_O$, the variance of the average m_X would be numerically the same as if the autocorrelation structure had been applied to the continuum. Thus, the correct solution can be obtained with greatly simplified computation. A similar approximate approach can be used in calculating size effect factors for 2-D and 3-D problems.

Spatial Extremes

The importance of spatial variability on calculated predictions depends both on the volume of soil influenced and on the mode of performance. For modes of performance which depend on average soil properties within a large volume of soil, spatial variability partially averages out, as described above. However, for modes which depend on worst condtions, for example sliding along a discontinuity or internal erosion in a dam, spatial variability is accentuated. In this latter case the size-effect factor may be greater than one, and an alteration may be caused to the mean. These cases are outside the scope of the present report.

Example: Embankment Stability

In the slope stability calculation of Plate 2 the influence of spatial averaging apppears appears as the size effect factor R=0.7, but was not discussed. Here we consider the influence of failure size.

The example in Plate 2 comes from a large water resource development involving low water-retaining dykes founded on soft clays. Three design cases were analyzed, as shown in Figure 27, these were a 6m single-stage dyke, a 12m single-stage dyke, and a 24m two-stage dyke. In the two-stage construction the foundation clays are allowed to consolidate for 12 months under a 12m fill which is then raised to 24m the following year. The worst case or design condition is end-of-construction, which is analyzed assuming undrained conditions. For illustrative purposes, only the analysis of the 24m dyke is presented here.

The principal uncertainties in the stability calculations are the undrained strengths of the foundation clays, the engineering properties of the embankment fill materials, and the geometry of the subsurface stratification. These are shown in Table 3, with their respective systematic and spatial variances.

The derivatives of factor of safety (F) with respect to the uncertain parameters in step 2 were calculated numerically using Simplified Bishop circular arc and Morgenstern-Price wedge-type failure geometries. For each design geometry a base-case analysis used all parameters at their means. This gives the best-estimate-F. For each parameter, additional calculations were made to numerically determine the derivative of F near the mean. The derivative was calculated as the ratio of change in F to change in input parameter, $\Delta F/\Delta x$, as shown in Plate 4.

The analysis of the first-stage 12m dyke shown in Plate 4 used circular arc failure surfaces and Simplified Bishop analysis. The square of the derivative dF/dx with respect to each principal uncertainty was multiplied by

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the corresponding variances of Table 3 to obtain the contribution of each uncertain input parameter to systematic and spatial uncertainty in the calculated value of factor of safety F. These are shown as variance contributions V_F. By Eqns. 14 and 25, the sum of these variance contributions over all the input uncertainties gives the overall spatial and systematic variances in the calculated value of F. From Plate 4, this total variance is $V_F=0.030$, and the corresponding standard deviation is $s_F=0.17$ ($=\sqrt{0.030}$).

The total variance of 0.030 reflects the uncertainty in soil properties from point to point. Actually, the critical failure wedge for the 25m base case has a length of about 180m. Thus, some averaging of the point to point variations of soil properties takes place over the failure surface. A size effect factor R must be determined to correct for this averaging, and this factor is multiplied by the spatial part of V_F to obtain an estimate of V_F for the whole failure surface.

Following the procedure outlined in "Statistical Analysis of Geotechnical Data" (Contract Report GL-87-1), numerical integration is used to determine the approximate extent of spatial averaging over the circular failure surface with 180m length. This led to the reduction factor $R = 0.04$ for the spatial component of variability. That is, the critical failure surface is sufficiently large that averaging across the surface reduces the point by point spatial variation by about 95%. Thus, the variance of F for the critical surface is,

$$V_F = R \; V_{F,\text{spatial}} + V_{F,\text{systematic}}$$
$$= 0.04 \; (0.030-0.012) + 0.012$$
$$= 0.013 \; . \tag{38}$$

This procedure is approximate. The variance composition relies on a linearization, which for stability analyses introduces little error. In addition, the analysis first minimizes F over trial failure surfaces and then performs the error analysis on that critical F surface. More is said about the choice of critical failure surface in Part IV.

Careful examination of Fig. 10 indicates that a relatively weak layer in the foundation clays occurs just at about the boundary of the marine and lacustrine clays. Plate 5 shows a calculation to check the stability of the embankment against failures which might pass through this layer. This calculation uses a wedge failure geometry in conjunction with the Morgenstern-Price method of analysis. The result is of considerable interest in that it illustrates the utility of an error analysis over simple factor of safety results. Note that the best estimate factor of safety for a failure wedge through the weak layer is only $m_p=1.24$, as compared to $m_p=1.45$ for the analysis with uses average clay properties. On the other hand, because the failure zone is well defined and because considerable, specific data were collected in the weak layer, the standard deviation associated with this calculation is comparatively small. Thus the reliability of the calculated result is actually greater than the corresponding averaged analysis. This reliability is measured by the reliability index β which is introduced in Part IV.

Other Methods for Error Analysis

The approach to propogating uncertainty through an engineering model used here is based on a first-order propogation of variance. This is a common technique and has a variety of names in the various disciplines to which it finds application. It is sometimes called "first-order second-moment" (FOSM)

analysis, and sometimes simply "error analysis." However, there are several other ways to analyze the effect of input uncertainties on output uncertainties. Among the more often encountered of these other methods in civil engineering practice are the point estimate method, the adjoint method, Monte Carlo simulation, and response surface techniques. The intent of this section is to breifly introduce these other methods and provide an introductory reference to their literature.

Point Estimate Method

The point estimate method, originally due to Rosenbleuth (1975), uses a limited number of deterministic calculations made at well-chosen sets of input parameter values to approximate the mean and standard deviation of a predicted variable. For example, in the simplest case of Eqn. 26 when both x and y are scalars, three deterministic calculations are made. These use as input, (a) the mean of x, (b) the mean plus one standard deviation of x, and (c) the mean minus one standard deviation of x. The calculated results are used to estimate a mean and standard deviation of y by the relations,

$$m_y \approx \frac{|\ g(m_x+s_x) + g(m_x-s_x)\ |}{2} \approx g(m_x)$$

$$s_y \approx \frac{|\ g(m_x+s_x) - g(m_x-s_x)\ |}{2}.$$

(39)

Similar techniques have been proposed for multivariate and correlated input.

The point estimate method gives exact results when g(x) is linear. Thus, in this particular case the point estimate and first-order technique give the same answer. They do not necessarily give the same answer when g(x) is nonlinear.

The point estimate method is convenient for many geotechnical uses, although the goodness of its approximation appears not to have been widely studied to date. Nonetheless, its use will probably become more widespread in the future.

Response Surface Techniques

Response surface techniques are related both to variance propogation and simulation, finding their most frequent use with models that are numerical, possibly implicit, difficult to analytically propogate variance through, and expensive to run. Response surfaces are in essence multivariate regression analyses. Multiple runs of the model are made in the vicinity of the mean of the input parameter values and a regression surface of chosen complexity is fit to the output predictions obtained. This regression surface is presumably less complicated than the model function itself, and yet can be taken as an approximation to which variance propogation or other techniques are applied. At the same time, many fewer runs of the model are made than with simulation, and thus cost is reduced. Response surface approaches are often applied to risk analysis problems associated with nuclear power and waste facilities, and to structural reliability problems (McCormick, 1981).

Adjoint Sensitivity Analysis

Adjoint techniques evaluate the proportionate effect of a perturbation in input parameters on the resulting perturbation in an output prediction. That is, they lead to an evaluation of the quantity $\{(\Delta y_j/\Delta x_i) \ x_i/y_j\}$, in which y_j

is the jth component of the prediction and x_i is the ith input parameter.
Adjoint techniques are conveniently applied to large numerical models
involving the solution of systems of linear equations. By manipulating the
linear algebra of such solutions, adjoint results can be obtained in the
course of computations. While adjoint techniques are usually used to obtain
sensitivies of a model rather than to perform quantitative uncertainty
analysis, the results can be used to numerically obtain derivatives, and thus
to provide the means for first-order variance propogation (Hadlock, 1984).

Monte Carlo Simulation

M.C. Simulation uses many repetitions of deterministic calculations in
which values of input parameters are randomly generated from specified
probability distributions. The result of simulation is a set of many
predictions of each output parameter which are treated as empirical data from
which statistical inferences of the means, variances, etc. of output
predictions can be made. An advantage of simulation is simplicity. It
requires none of the mathematics of variance propogation, adjoint analysis, and
related techniques. On the other hand, simulation has three important
limitations. It is expensive because the deteministic model must be run many
times. For example, at least several hundred trials are typically needed. It
requires not only means and variances of input parameters, but entire
probability distributions. These may be ambiguous or arbitraty. Finally, the
components of uncertainty are lumped together in simulations. Thus, differing
effects are hard to unravel. Nevertheless, simulation is an important tool
when a model is complicated, involves logical branching, or on other occasions
when variance propogation and related techniques cannot be used (Hammersley and
Handscomb, 1964).

PLATE 1

SUBJECT: Bearing capacity of a shallow footing.

PROBLEM

$q_v = 10$ kip

$\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$

b=5'

D=0'

///////////////////////////////////////
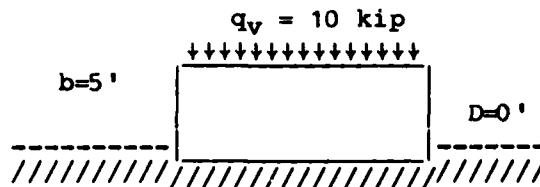
SOIL PROPERTIES

$m_{\phi'} = 36.4°$ (n=5)

$V_{\phi'} = (1.14°)^2$

$\gamma = 120$ pcf

(a) BEST ESTIMATE (MEAN) OF BEARING CAPACITY

$N_\gamma \cong 0.01 e^{0.25\phi'}$

$q_v = (1/2)\ \gamma\ B\ N_\gamma$

$m_{qv} = (1/2)\ (120)\ (5)\ m_{N\gamma}$
$= (1/2)\ (120)\ (5)\ (89.6)$
$= 26.9$ kip

(b) SPATIAL VARIABILITY OF BEARING CAPACITY

$V_{N\gamma} \doteq (dN_\gamma/d\phi')\ V_{\phi'}$

$V_{\ln N\gamma} \doteq (0.25)^2\ V_{\phi'}$
$= (0.25)^2 (1.140)^2$
$= (0.29)^2$

$V_{N\gamma} \doteq (\dfrac{dN_\gamma}{d\ln N_\gamma})^2\quad V_{\ln N\gamma}$
$= 89.6)^2\ (0.29)^2$
$= 26.0$

$V_{qv} \doteq (\dfrac{dq_u}{dN_\gamma})^2\quad V_{N\gamma}$
$= [(1/2)(120)(5)]^2\ (26.0)^2$
$= (7.8\ \text{kip})^2$

PLATE 2   Page 1/2

SUBJECT: 2-D slope stability analysis 1-berm case

Problem:
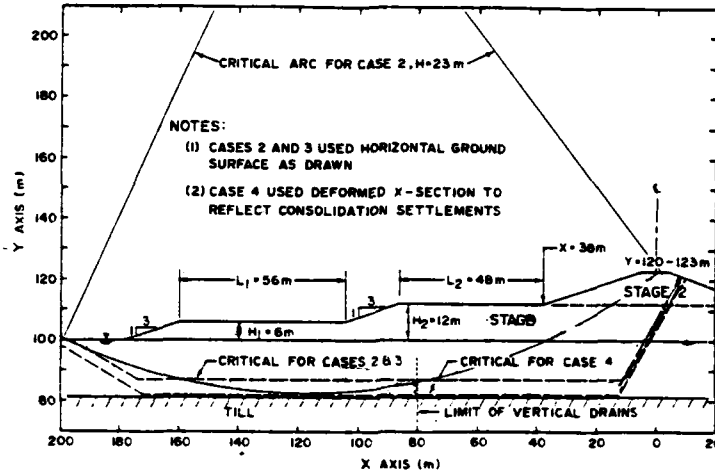
(a) BEST ESTIMATE (EXPECTED VALUE) OF FS AGAINST INSTABILITY, 6-m CASE

$$\overline{FS} = \underline{1.84} \quad \text{[by modified Bishop method]}$$

(b)   UNCERTAINTY (VARIANCE) IN FACTOR OF SAFETY

$$V[FS] = \sum_i \left(\frac{\partial FS}{\partial x_i} \right)^2 V[x_i]$$

uncertainty in soil parameters
assumed mutually independent.

$$\left(\frac{\partial FS}{\partial x_i} \right)$$

shown on table,
calculated numerically.

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PLATE 2   Page 2/2

SUBJECT: 2-D slope stability analysis 1-berm case

Parameter	$\Delta FS/\Delta x_i$	Variance Systematic	TOTAL	$(\Delta F/\Delta x_i)^2 \cdot V(x_i)$ Systematic $\times 10^{-4}$	TOTAL
ϕ'	0.00221	3.0	4.0	nil	nil
γ_{FILL}	0.068	1.0	2.0	46	92
D_{crust}	nil	0.036	1.0	nil	nil
D_{till}	0.010	1.0	1.0	1	1
$c_u(L)$	0.0258	24.9	99.7	166	664
$c_u(M)$	0.020	7.6	47.6	30	190

$$V[FS] \quad = \quad 0.0243 \quad 0.0947$$
$$SD[FS] \quad = \quad 0.156 \quad 0.308$$
$$V_{30\phi}[FS] @ R=0.7 \quad = \quad 0.0243 + 0.0493$$
$$= \quad 0.0736$$

(c) RELIABILITY INDEX

$$\beta = \frac{\overline{FS} - 1.0}{s_{FS}}$$

$$= \frac{1.500 - 1.0}{\sqrt{0.0736}}$$

$$= 1.84$$

PLATE 3

SUBJECT: Spatial Averaging of SPT Blow Count Data

BORING #	1	2	3	4	5	6
DEPTH						
1	2	1	2	8	3	4
2	3	8	5	3	7	4
3	8	6	5	3	7	5
4	6	6	7	0	8	7
5	0	2	5	2	5	0
6	3	2	4	1	9	4
7	3	5	0	0	4	1
8	8	3	0	8	8	7

Average and standard deviation of average of n=1,2, and 8 data:

n = 1 m_N = 3.3
          s_m = 2.5

n = 2 m_N = 5.7
          s_m = 2.0

n = 8 m_N = 4.2
          s_m = 0.9

number of N-values averaged

PLATE 4   Page 1/2
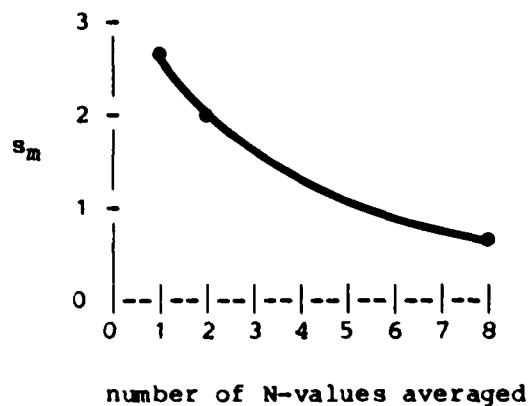
SUBJECT: 2-D slope stability analysis
           24m case, circular arc analysis

Problem:
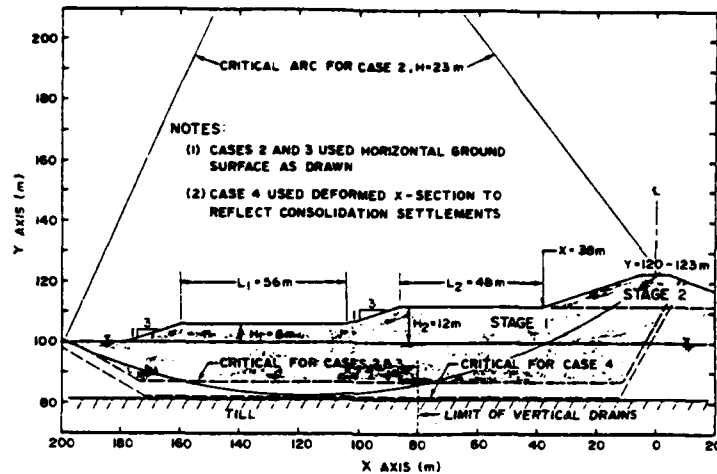
(a) BEST ESTIMATE (EXPECTED VALUE) OF FS AGAINST INSTABILITY, 24-m CASE

$$\overline{\overline{FS}} = \underline{1.43}$$ [by Simplified Bishop method]

(b) UNCERTAINTY (VARIANCE) IN FACTOR OF SAFETY

$$V[FS] = \sum_i \left(\frac{\partial FS}{\partial x_i} \right)^2 \ V[x_i]$$ uncertainty in soil parameters
assumed mutually independent.

$$\left(\frac{\partial FS}{\partial x_i} \right)$$ shown on table,
calculated numerically.

PLATE 4   Page 2/2

SUBJECT: 2-D slope stability analysis 24m case

Parameter	$\Delta FS/\Delta x_i$	Variance Systematic	TOTAL	$(\Delta FS/\Delta x_i)^2 \cdot V(x_i)$ Systematic x 10^{-4}	TOTAL
Intact Clay					
$\tau(M)$	0.0018	13.3	41.5	nil	1
$\tau(L)$	0.012	26.3	88.5	38	127
Consolidated Clay					
$\tau(M)$	0.0021	52.7	111.5	2.3	4.9
$\tau(L)$	0.009	62.0	124.4	50	101
ϕ	0.0088	3.0	4.0	2	3
YFILL	0.055	1.0	2.0	30	61

$$V[FS] = 0.0122 \quad 0.0298$$
$$SD[FS] = 0.111 \quad 0.173$$
$$V_{ave}[FS] @ R=0.04 = 0.0122 + 0.0012$$
$$= 0.0134$$

(c) RELIABILITY INDEX

$$\beta = \frac{\overline{FS} - 1.0}{s_{FS}}$$

$$= \frac{1.43 - 1.0}{\sqrt{0.0134}}$$

$$= 3.68$$
====

PLATE 5   Page 1/2

SUBJECT: 2-D slope stability analysis
           24m case, weakest layer 180m

DATE: _____   REFERENCE: _____

Problem:
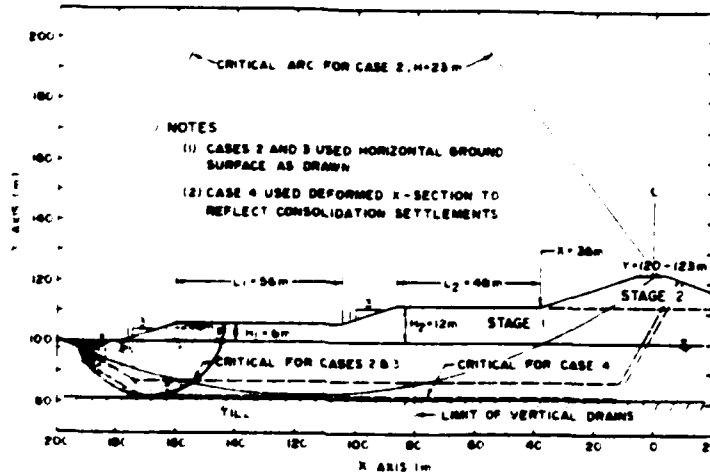
(a) BEST ESTIMATE (EXPECTED VALUE) OF FS AGAINST INSTABILITY

$$\overline{\overline{FS}} = \underline{1.24} \quad \text{[by Morgenstern Price Method]}$$

(b) UNCERTAINTY (VARIANCE) IN FACTOR OF SAFETY

$$V[FS] = \sum_i \left(\frac{\partial FS}{\partial x_i} \right)^2 V[x_i]$$

uncertainty in soil parameters
assumed mutually independent.

$$\left(\frac{\partial FS}{\partial x_i} \right)$$

shown on table,
calculated numerically.

PLATE 5   Page 2/2

SUBJECT: 2-D slope stability analysis 25m case, weakest layer 180m

Parameter	$\Delta FS/\Delta x_i$	Variance Systematic	TOTAL	$(\Delta FS/\Delta x_i)^2 \cdot V(x_i)$ Systematic	TOTAL x 10^{-4}
ϕ'	0.005	3.0	4.0	1	1
γ_{FILL}	0.048	1.0	2.0	23	46
D_{crust}	0.007	0.036	1.0	Nil	Nil
D_{till}	0.0	1.0	1.0		
$c_u(L)$	0.028	Cov(β)≈0.15	5.6	23	44
$c_u(M)$	0.013	7.6	47.6	13	80

$$V[FS] = 0.0060 \quad 0.071$$
$$SD[FS] = 0.0775 \quad 0.131$$
$$V_{ave}[FS] @ R=0.04 = 0.0060 + 0.0022$$
$$= 0.0082$$

(c) <u>RELIABILITY INDEX</u>

$$\beta = \frac{\overline{FS} - 1.0}{s_{FS}}$$

$$= \frac{1.24 - 1.0}{\sqrt{0.0082}}$$

$$= \underline{2.69}$$
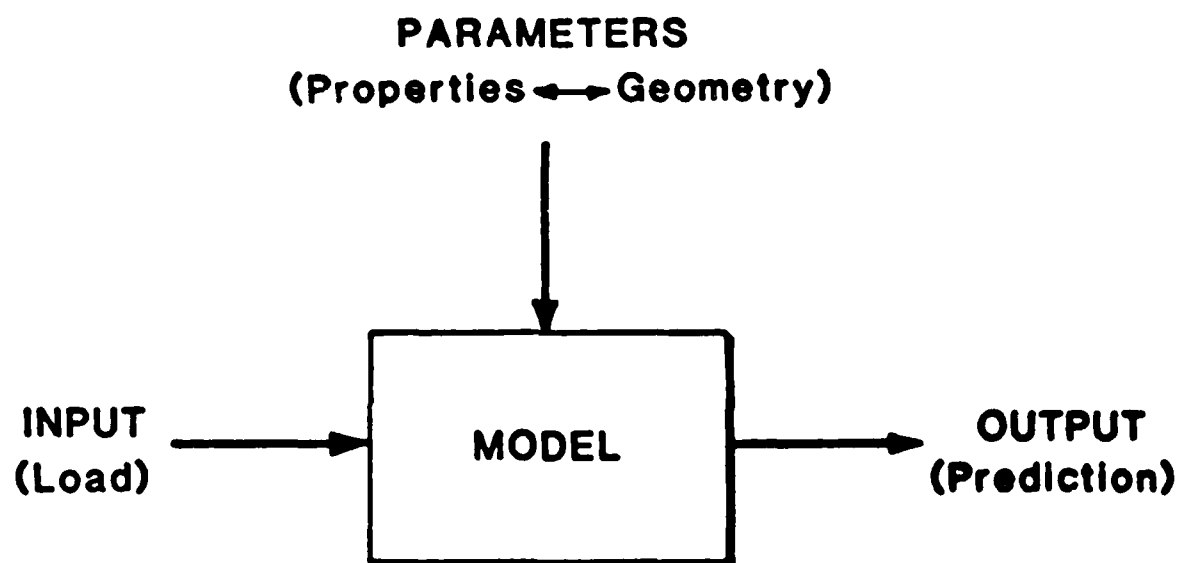
Figure 25.   Error Propagation Through the Model $y = q(x)$.
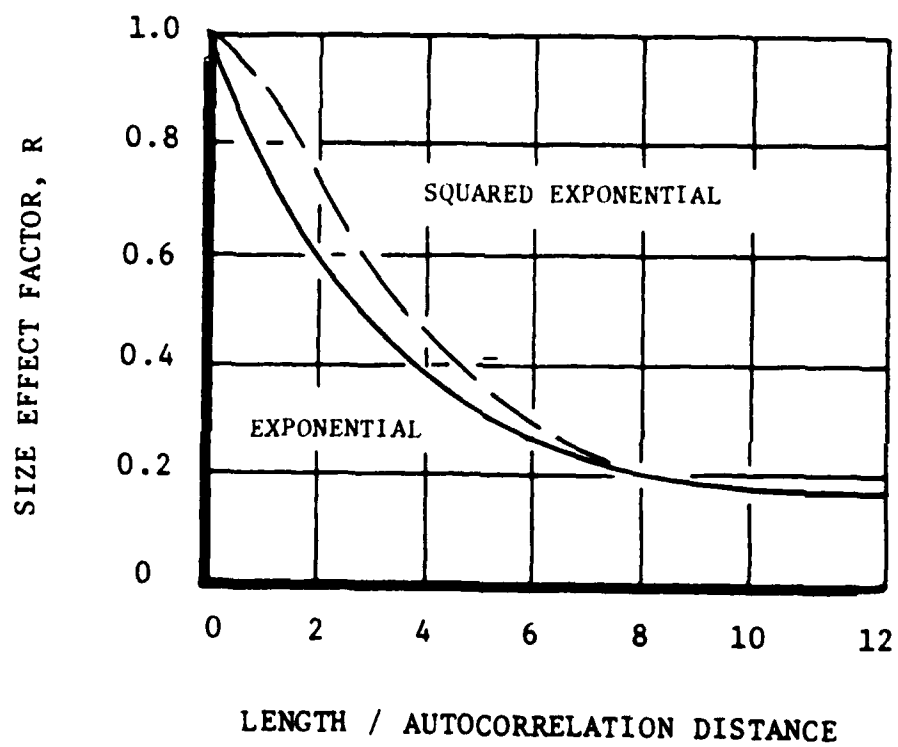
Figure 26. Spatial Averaging in One-Dimension
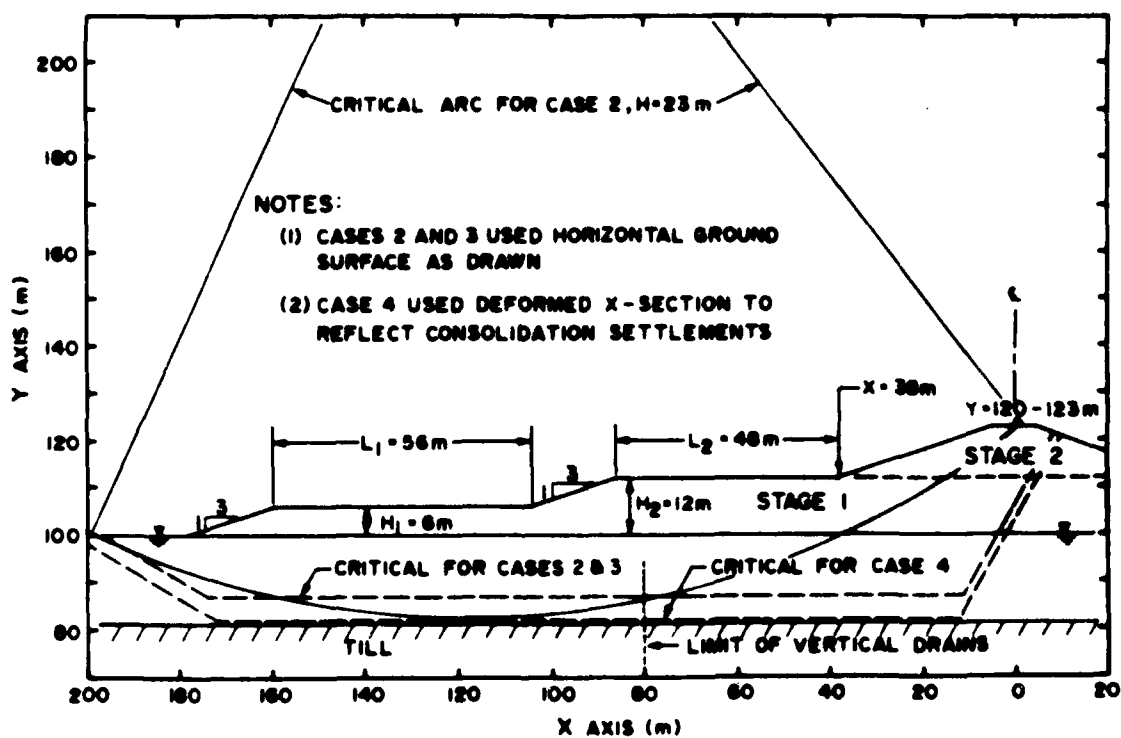
Figure 27.   Design Cases for Embankment Stability Analysis.

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PART IV: THE RELIABILITY INDEX

In traditional geotechnical analysis the adequacy of a design is expressed by a factor of safety, defined variously as the ratio of capacity to demand,

$$F = \frac{capacity}{demand} \qquad (40)$$

The factor of safety makes no allowance for uncertainty.

When performance is predicted by both a best estimate and a measure of uncertainty, a more complete safety index can be used. One index which combines both best estimate and uncertainty is the 'reliability index', β. In essence, β measures the number of standard deviations separating the best estimate of performance from some unacceptable or 'failure' value. Part IV of this report defines the reliability index β, shows its relationship to factor of safety F, and gives examples of its use.

Definition of the Reliability Index β

An error analysis translates the effect of uncertainties in engineering properties to uncertainties on calculated results. Using any of the techniques of Part III--except Monte Carlo simulation--uncertainties in engineering properties are described simply by their means, standard deviations, and correlations with other properties. These means, standard deviations, and correlations are translated to means, standard deviations, and correlations for calculated predictions.

Such a description of uncertainty contains more information than can be incorporated in a factor of safety, yet does not allow so-called probabilities of failure to be determined (i.e., probabilities that the actual performance of

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a facility is inadequate compared to some defined limit state). Determining

probabilities of failure requires information on the distribution of

probability over the predicted performance y, and no such assumption is made.

Instead, the mean and standard deviation of y are combined in an index, β that

describes reliability as the number of standard deviations seqarating the best

estimate of y from its defined failure value y_0,

$$\beta = \frac{m_y - y_0}{s_y} = \text{"reliability index"} \tag{41}$$

Lower values of β imply lower reliability. $\beta=0$ means the best estimate of

performance equals the failure criterion, that is $m_y=y_f$. $\beta>0$ means that $m_y>y_f$,

because the standard deviation is always positive. Typical β's for current

geotechnical design range from 2 to 4.

The reliability index is a useful measure because it balances the safety

implied by a best estimate against the uncertainty in that estimate. Thus, β

can distinguish between, (i) a high mean factor of safety with correspondingly

high uncertainty, and (ii) a low mean factor of safety with correspondingly low

uncertainty. In Fig. 28, Design #1 has a higher mean factor of safety (F)

than design #2, but also a larger standard deviation. Since the probability of

inadequate performance is related to the area under the frequency distribution

to the left of F=1.0, although design #1 has a higher mean F it also has a

greater likelihood of performing inadequately. The reliability index captures

this distinction.

The reliability index β is a direct measure of the reliability of a

calculation with respect to facility performance. β measures the confidence

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one should have that errors in a calculation are small relative to the margin of safety separating a best estimate from some design limit.

The reliability index β is not a statement of probability. However, in some situations it is convenient to relate β to a _nominal_ probability of failure, p_f, using a Normal (i.e., bell-shaped) distribution of uncertainty on y (Fig. 29). For β less than 2 or 2.5, nominal p_f is insensitive to the assumed shapae of the distribution of probability over y; however, for β larger than 2.5 nominal p_f is sensitive to the distribution chosen.

Limitations of β

The reliability index β defined by Eqn. 41 is useful but has limitations. The most important limitations of β are,

1. β can be load path dependent.

2. β is not invarient to certain mechanically equivalent mathematical transformations of the definition of failure.

Each of these limitations also applies to the factor of safety F.

The load-path dependence of β can be illustrated by considering the bearing capacity of an unembedded footing subject to an inclined load (Fig. 30). The combinations of vertical load V and horizontal load H which define the limiting conditions for loads on the footing form a curved envelope. Combinations of V and H inside the envelope can be resisted by the footing; combinations outside cannot.

Starting from initial load P_o, two possible load-paths to failure are, (1) increasing the horizontal load H until the failure envelope is reached at P_1, and (2) increasing the vertical load V until the failure envelope is reached at P_2. For the horizontal load path the factor of safety is

$$F_H = \frac{H_2}{H_1} \tag{42}$$

For the vertical load path the factor of safety is

$$F_V = \frac{V_2}{V_1} \tag{43}$$

Depending on the location of P_O within the safe region, the factors of safety F_H and F_V can be arbitrarily changed independent of one another. Only in exceptional situations will F_H and F_V be the same. Exactly the same load-path dependence applies to the respective reliability indicies β_H for horizontal loading and β_V for vertical loading.

The noninvariance of β can be illustrated by considering a rock block sitting on an inclined slope with asperities (fig. 31). Depending on how the forces are resolved, the factor of safety against sliding can be stated as,

$$F_1 = \frac{\tan \phi}{\tan(\theta-i)} \quad ; \tag{44}$$

or as,

$$F_2 = \frac{\tan (\phi+i)}{\tan(\theta)} \quad . \tag{45}$$

Arbitrarily choosing parameter values, let $\theta=25°$ and $i=5°$, both known. Let the best estimate of ϕ be $m_\phi=40°$ and the standard deviation be $s_\phi=3°$. Then, the mean factors of safety and β's corresponding to Eqns. 44 and 45 become,

$$m_{F_1} = 2.31 \qquad \beta_1 = 5.31 \tag{46}$$

$$m_{F2} = 2.14 \qquad \beta_2 = 5.08 \tag{47}$$

Advantages of the Reliability Index β

Error analysis of the type presented in this report is in essence a replacement for traditional sensitivity analysis approaches to evaluating the reliability of engineering calculations. The reliability index β is a convenient single measure which combines the two results of an error analysis, namely the best estimate and the standard deviation.

In practice, the reliability index β has important advantages for engineering analysis and design. First, it provides a traceable path through a set of calculations by which uncertainties are accounted for. The principal benefit of this traceable path is quality assurance. Second, it provides a means for explicitly incorporating the extent of information gathering with the reliability of an engineering prediction. The benefit of tying information to reliability is that the quantitative basis of predictions can be demonstrated. Third, it provides a means for increasing the consistency of design decisions. The benefit is that the conservatism of design, reflected in design factors of safety, can be balanced against the confidence of a prediction. Each of these three capabilities expands the engineer's ability to produce a quality design.

Quality Assurance

Quality assurance is a must in civil engineering. Unlike many branches of engineering, the civil engineer typically designs and builds a single copy of his product. The concept of an acceptable failure rate, which is widespread in manufacturing and electronics, is foreign to the dam designer or bridge engineer. The means for assuring quality in engineering is explicitness. Error analysis and β-values provide that explicitness by forming an accounting

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sheet for uncertainties.

Relation of Data to Confidences

Using factors of safety, the influence of site characterization data on reliability and on design factors of safety is implicit. There is no quantitative tie-in between information and confidence in performance predictions. The β index provides a means for quantifying the effect of information on confidence, in that information influences the standard deviation of a prediction and thus β.

Consistent Factors of Safety

As shown schematically in Fig. 28, the reliability of a prediction of engineering performance is not completely described by F alone. A high mean factor of safety combined with a high standard deviation may be a less reliable prediction than a low mean F combined with a low standard deviation. The reliability index, β, captures this distinction.

The choice of design factors of safety for different conditions can be made consistent, in that the same level of reliability is implied, by setting the corresponding reliability indices equal. As shown in Fig. 32, for two design cases in which the coefficients of variation of calculated F's are 0.10 and 0.20, respectively, the β's corresponding to expected design F's of 1.5 are different:

Mean Factor of Safety

β	coefficient of variation	
	$\Omega_F = 0.10$	$\Omega_F = 0.20$
1.50	1.18	1.43
2.00	1.25	1.67
3.00	1.43	2.50

To obtain F's implying consistent reliability, at say $\beta=2$, the target values of mean factor of safety would have to be different in the two cases. Thus, error analysis provides a vehicle for improving internal consistency of criteria or standards for design.

Example: Bearing Capacity of a Shallow Footing

For the footing described in Plate 1, uncertainty in the soil friction angle caused by data scatter led to a best estimate (mean) of bearing capacity of 26.9 ksf and a standard deviation of 7.8 ksf. If the design stress were 10 ksf what would be the reliability index β for the calculation? Applying Eqn. 4',

$$\beta = \frac{m_{qv} - q_{vo}}{s_{qv}}$$

$$= \frac{26.9 \text{ ksf} - 10 \text{ ksf}}{7.8 \text{ ksf}} \tag{48}$$

$$= 2.2$$

Example: Footing Settlement on Sand

Plate 6 shows a settlement calculation for shallow footings on appoximately ten meters of uniform wind-blown sand. The facility is an industrial plant founded on a large number of footings. The site was characterized by SPT borings, predictions of settlement were made based on the N values, and settlements were subsequently measured.
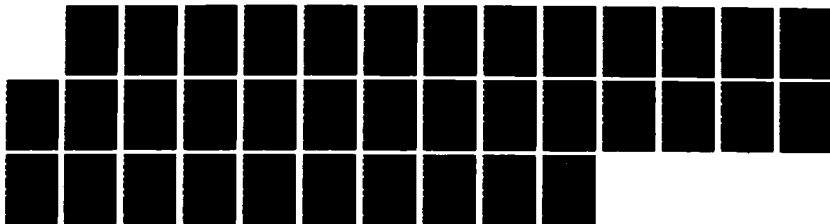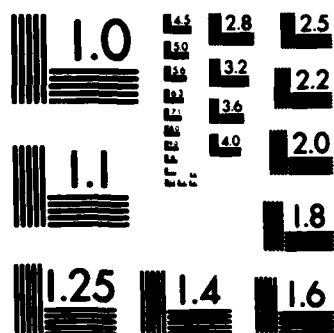
MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

Calculating Footing Settlement

Footing settlement can be predicted by any of a number of equations. Peck and Bazaara's equation is a modification of the Terzaghi and Peck envelope,

$$\rho = \left(\frac{2\Delta q}{m_N}\right) \left(\frac{2b}{1+b}\right)^2 \left[1 - \frac{1}{4}D/b\right] \quad , \tag{49}$$

in which ρ is settlement (inches), Δq is allowable applied stress (TSF), m_N is (vertically) averaged corrected blow count, and b is footing width (ft). Water table elevation is ignored. The term involving D/b, where D=embedment depth is a depth correction factor. In the present case $D/b=0.5$. For square footings of design width $b=10'$, the best estimate of ρ at the allowable stress of 3TSF (6ksf) is shown in Plate 6.

Spatial Component of Settlement Uncertainty

The variance of ρ due to uncertainty in m_N is calculated by noting that ρ is inversely proportional to m_N. Therefore, from Eqn. 35,

$$\Omega_\rho \doteq \Omega_{m_N} \tag{50}$$

m_N is the average blow count within a depth b equal to the footing width of the footing and thus its variance and coefficient of variation are less than those of the point by point blow counts, N. For this site blow counts are taken every 5 feet, thus the coefficient of variation of the vertically averaged blow counts was $\Omega_{m_N}=0.44$. If the distinction between real soil variability and measurement noise is ignored, the coefficient of variation of settlement ρ that one would predict from the data scatter should be about 0.44. That is, ignoring noise in the data one would expect the spatial variability among

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footing settlemtns to have $\Omega_\rho = 0.44$. Alternately, Eqn. 30 could have been used to find the same result with more effort.

In comparison, the observed values of total settlement for 268 footings at this site have a mean of about 0.35", and a standard deviation of 0.12". Thus, the observed variability has a coefficient of variation $\Omega_\rho = 0.34$, less than one might expect based on the scatter in the N-values.

The discrepancy, of course, is caused by measurement noise, which must be removed from the data scatter before an accurate estimate of spatial variability in the footing settlements can be obtained. The noise content of the data is estimated from the autocovariance function (Fig. 33) to be about 50% of the data scatter variance. This means that,

$$\Omega_{soil})^2 = (\Omega_{data})^2 (0.5) = (0.44)^2 (0.5)$$
$$= (0.31)^2$$

(51)

which is close to the $\Omega_\rho = 0.34$ observed in the settlements records.

Systematic Component of Settlement Uncertainty

In addition to spatial variability, the limited number of borings causes statistical error in the prediction of average settlement. With 50 borings and hence 50 SPT measurements at any elevation, the statistical error in the estimated mean blow count at any elevation in the upper levels is

$$Vm_N \doteq V_N / 50 \qquad .$$

(52)

This reflects uncertainty on the average settlement of all the footings at the site.

The settlement model itself introduces bias which differs from site to site. For Eqn. 49 comparison data of predicted vs. observed settlements yield a mean bias (Fig. 34) of $m_b=1.46$ and a standard deviation of $s_{Bb} = 1.32$, in which b=observed settlement/predicted settlement. Correcting the earlier estimate for this model bias,

$$m_\rho' \doteq m_b \, m_\rho \qquad , \qquad (53)$$

in which m_ρ, is the corrected mean settlement. The variance of the corrected settlement is found using Eqn.32 as,

$$V_\rho \doteq V_b \, m_\rho^2 + m_b^2 \, V_\rho \qquad (54)$$

The poor correlation of the settlement model to actual footing performance introduces a large model error if data are unavailable for calibrating the model to a particular site. This model error is difficult to divide into scatter and systematic parts because data of the sort used to calibrate models are mixtures from many sites and model tests. However, the calculations in Plate 6 attest to the importance of model uncertainty in settlement predictions.

In service, the footings were exposed only to 40 to 70% of the allowable load used for predicting settlements. Also, footing dimension and embedments varied. Therefore, the mean predicted settlement and the mean observed are not comparable. However, because Eqn. 49 is multiplicative, Ω_ρ should be unaffected by these differences.

Example: Final Consolidation Settlement

Javette (1983) applied a similar approach to the problem of consolidation settlement of San Francisco Bay Mud under a uniform surcharge. While his analysis differs in notation and somewhat in form from the methods developed here, in principle it is the same. For this illustration, however, Javette's analysis is slightly rearranged to fit the present format.

In calculating the mean and standard deviation of total settlement at the end of primary consolidation, Javette makes several assumptions: (1) unit weights are known, (2) compression and recompression ratios are described by means, variances and covariances, and (3) the trend of overconsolidation (i.e., a function of maximum past pressure) with depth can be approximated by a smooth mathematical curve. Then, settlement is calculated by dividing the Bay Mud stratum into k hypothetical layers with thickness twice the vertical autocorrelation distance, and summing the settlement of each layer to obtain,

$$\rho = \sum_{i}^{k} \log(\sigma_{vm}'/\sigma_{v0}') \, CR_{\epsilon,c} + \log(\sigma_{vf}'/\sigma_{vm}') \, CR_{\epsilon,r} \qquad (55)$$

in which σ_{v0}', σ_{vm}' and σ_{vf}' are the in situ vertical effective stress, maximum past pressure, and final vertical effective stress, all in layer i, respectively; $CR_{\epsilon,r}$ is the recompression ratio, and $CR_{\epsilon,c}$ the virgin compression ratio. The mean settlement is obtained by substituting the means of σ_{vm}', $CR_{\epsilon,r}$, and $CR_{\epsilon,c}$ into the equation, σ_{v0}' and σ_{vf}' being assumed known.

The components of uncertainty adopted by Javette are shown in Table 4. Thus, the variance of end of primary settlement is calculated from Eqn. 31 as shown in Plate 7. The layer thickness are chosen using the trick described in

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Part III, so the variations of properties among layers can be taken as independent and R=1.

To compare predicted standard deviations with the range of settlements observed in the field, the rate of consolidation had to be incorporated in the calculations. Some of the field measurements were taken at times prior to end of primary compression, and some included secondary compression. For this purpose the coefficient of consolidation c_v was assumed known and the coefficient of secondary compression was described by a mean and variance. In fact, c_v could also be described as an uncertain parameter, and in many consolidation problems it is an important source of uncertainty. Javette, reasoning on the basis that 95% of the area under a Normal (bell shaped) distribution lies within ±1.96 standard deviations of the mean, used $\pm(1.96s_\rho)$ as a predicted range of settlement. This range is shown in Fig. 34 comparing favorably with the field observations.

```
=============================================================

          Table 4: Variance Assumptions for 1D Settlement

=============================================================
Variance Component     Contribution
=============================================================


Spatial variation   --   equal to the data scatter of the
                         consolidation parameters $C_{\epsilon,c}$, $C_{\epsilon,r}$.
-------------------------------------------------------------
Measurement noise   --   unknown and neglected.
-------------------------------------------------------------
Systematic error    --   statistical error in $C_{\epsilon,c}$ and $C_{\epsilon,r}$
                         due to limited numbers of tests.


                    --   differences in the maximum past
                         pressure profile from one location
=============================================================
```

PLATE 6   Page 1/3

SUBJECT: Calculation of footing settlement on sand.

PROBLEM

$\Delta q = 3$ TSF

B=10'

D=5'

allowable $\rho = 1"$
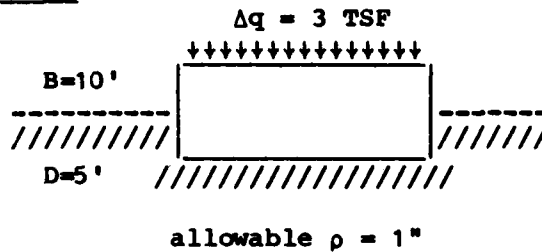
SOIL PROPERTIES

m_N = vertically averaged
SPT blow count, corr'd.
= 25 bpf

$s_N = 11$ bpf

$\Omega_N = 11/25 = 0.44$

(a) BEST ESTIMTE (MEAN) OF SETTLEMENT

$$m_\rho \doteq \left(\frac{2\Delta q}{m_N}\right) \left(\frac{2B}{1+B}\right)^2 [1 - \frac{1}{4} D/B] = \left(\frac{2 \cdot 3}{25}\right) \left(\frac{2 \cdot 10}{1+10}\right)^2 [1 - \frac{1}{4} 5/10]$$

$$= 0.70"$$

$m_\rho' \doteq m_b \, m_\rho$, in which b = model bias correction as in Equation 53

$$= (1.46)(0.70") = 1.02"$$

(b) UNCERTAINTY (VARIANCE) OF SETTLEMENT

Spatial Variability
from Eqn. 35, $\Omega_\rho \triangleq \Omega_{mN}$

$$\Omega_\rho \doteq \Omega_N = \frac{\sqrt{V[N]}}{m_N} = \frac{\sqrt{V[N]/2}}{m_N} = \frac{\sqrt{11^2/2}}{25} = 0.32$$

$$V_\rho = (\Omega_\rho \, m_\rho)^2 = [(0.32)(0.70")]^2 = (0.22")^2$$

$$V_{\rho'} = (\Omega_\rho \, m_{\rho'})^2 = [(0.32)(1.02")]^2 = (0.32")^2$$

PLATE 6   Page 2/3

SUBJECT: Calculation of footing settlement on sand.

((b) UNCERTAINTY (VARIANCE) OF SETTLEMENT)

Systematic Error

Statistical Estimation Error:

$$\Omega_{mN}^2 = \Omega_N^2/n = (0.32)^2 / 50 = (0.05)^2$$

$$\Omega_\rho = \Omega_{mN} = (0.05)^2$$

$$V_\rho = (\Omega_\rho \ m_\rho)^2 = [(0.05)(0.7)]^2 = (0.04)^2$$

$$V_{\rho'} = (\Omega_\rho \ m_{\rho'})^2 = [(0.05)(1.02)]^2 = (0.05)^2$$

Model Bias: $V_{\rho'} = V_B \ m_\rho^2 + m_B^2 \ V_\rho$

$$= (1.32)^2 (0.70'')^2 + (1.46)^2 (0.31 \times 0.70'')^2$$

$$= 0.85 + 0.10 = (0.98)^2$$

(c) RELIABILITY INDEX

Spatial Variability Alone

$$\beta = \frac{m_\rho - \rho_o}{s_\rho} = \frac{|0.70'' - 1''|}{0.22''} = 1.36$$

PLATE 6   Page 3/3

SUBJECT: Calculation of footing settlement on sand.

Spatial Variability + Systematic Error

Total uncertainty = spatial variability + systematic error

$$V_\rho = (0.22)^2 + (0.04)^2 = (0.23)^2$$

$$V_{\rho'} = (0.32)^2 + (0.05)^2 + (0.98)^2 = (1.0)^2$$

without model uncertainty:

$$\beta_\rho = \frac{\rho 0 - m_\rho}{s_\rho} = \frac{1.0'' - 0.7''}{0.23''} = 1.30$$

with model uncertainty:

$$\beta_{\rho'} = \frac{m_{\rho'} - \rho 0}{s_{\rho'}} = \frac{1.0'' - 1.02''}{0.23''} = -0.09$$

PLATE 7   Page 1/2

SUBJECT: One dimensional consolidation settlement.

SITE CONDITIONS: The site lies on the eastern shore of San Francisco Bay
where in recent years a uniform layer of cohesionless fill
has been placed over Bay Mud . The Bay Mud systematically
varies in thickness across the site, and is presumed to be
free draining at both top and bottom. A large number of
consolidation tests were performed on samples of Bay Mud
from the site.

PROBLEM: Calculate 1-D final consolidation settlement under a 2m surcharge.

(a) BEST ESTIMATE (MEAN) SETTLEMENT

$$\rho = \sum_{i}^{n} H_i \{ \log(\sigma_{vm}'/\sigma_{vo}') \cdot C_{\epsilon,r} + \log(\sigma_{vf}/\sigma_{vm}') \cdot C_{\epsilon,c} \}$$

layer	H_i $\log \Delta\sigma_v'$ $\cdot C_{\epsilon,r}$	H_i $\log \Delta\sigma_v'$ $\cdot C_{\epsilon,c}$
1	0.035	--
2	0.034	0.023
3	0.007	0.110
4	--	0.151
5	--	0.128
6	--	0.112
7	--	0.100
8	--	0.090
9	--	0.083
10	--	0.076
11	--	0.070

$$m_\rho = 1.02m$$

The first column in the table gives layer number in a vertical profile;
the second gives layer deformation due to recompression from σ_{vo}' to a
σ_{vf}' less than σ_{vm}', or to σ_{vm}' ; the third gives layer deformation due
to virgin compression from σ_{vo}' to a σ_{vf}' greater than σ_{vm}', or from
σ_{vm}' to σ_{vf}'.

SUBJECT: One dimensional consolidation settlement.

(b) UNCERTAINTY (VARIANCE) OF SETTLEMENT

Spatial Variability, V_1:	layer	$\dfrac{H_i^2 \times 10^{-4}}{\log^2 \Delta \sigma_v'}$ $V[C_{\epsilon,r}]$	$\log^2 \Delta \sigma_v'$ $V[C_{\epsilon,c}]$
The second and third	1	0.616	--
columns, respectively, show	2	0.581	0.70
the variances of individual	3	0.246	1.61
layer deformations due to	4	--	2.98
uncertainties in soil	5	--	2.19
parameters. The sum of the	6	--	1.68
layer variances is the	7	--	1.34
total variance in	8	--	1.10
settlement prediction.	9	--	0.91
	10	--	0.77
	11	--	0.66

$$V_1[\rho] = 1.54 \times 10^{-3} m^2$$

Uncertainty in σ_{vm}':

$$V(\rho) = \sum_{i=1}^{3} \left(\partial \rho / \partial \sigma_{vm}'\right)^2 V(\sigma_{vm}') = \sum_{i=1}^{3} \left((H_i/\sigma_{vm}')(C_{\epsilon,r}-C_{\epsilon,c})\right)^2 V(\sigma_{vm}')$$

$$= 1.28 \times 10^{-3} m^2$$

Statistical Error, V_3:

$$\Omega_{1+2}(C_{\epsilon,.}) = 0.128$$

$$\Omega_3(C_{\epsilon,.}) = (0.128)/(\sqrt{n}=32) = 0.023 = \Omega_3(\rho)$$

$$V_3(\rho) = [\Omega_3(\rho) \; E(\rho)]^2 = [(0.023)(1.02")]^2 = 0.55 \times 10^{-3} m^2$$

Variance Composition:

component:	contribution:
spatial variability	1.54×10^{-3}
uncertainty in σ_{vm}'	1.28×10^{-3}
statistical error	0.55×10^{-3}

$$\text{TOTAL Variance} = 3.37 \times 10^{-3} m^2$$
$$s_\rho = 0.058 m$$
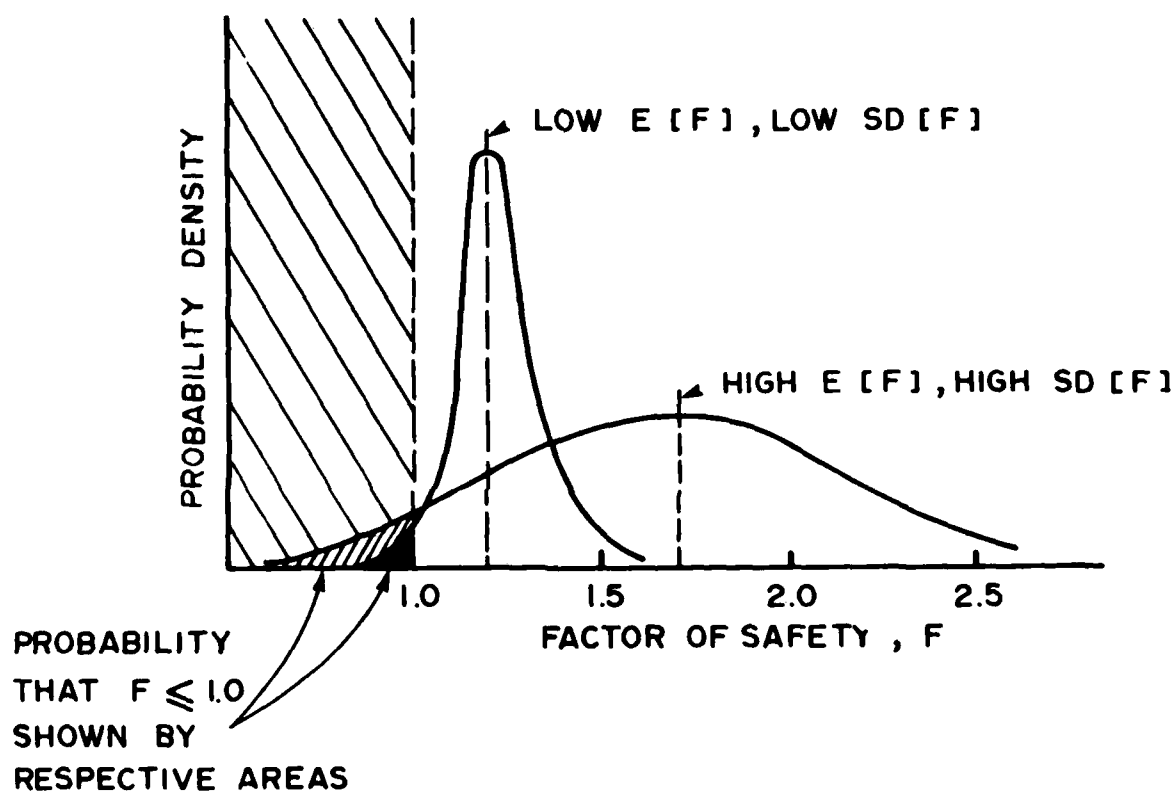$$\Omega_\rho = 0.058/1.02 = 0.079$$

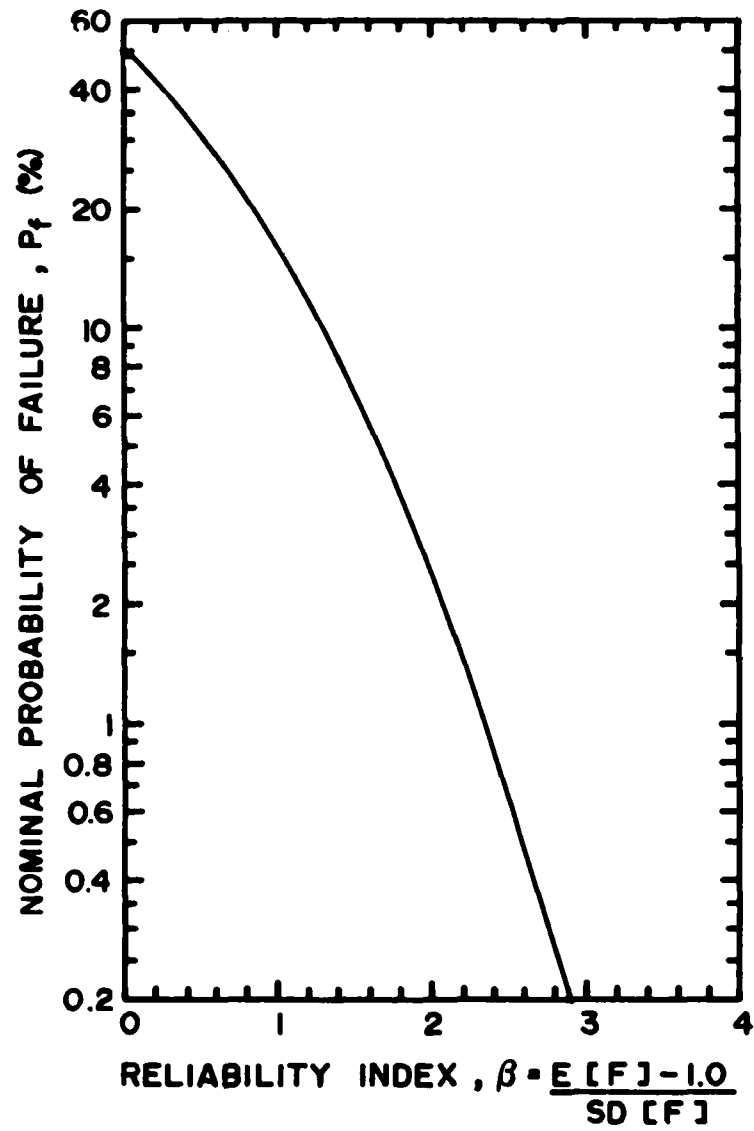Figure 28. Comparison of Two Calculations of Factor of Safety.

Figure 29. Relation of Reliability Index to Nominal Probability of Failure, Based on Normal Distribution.
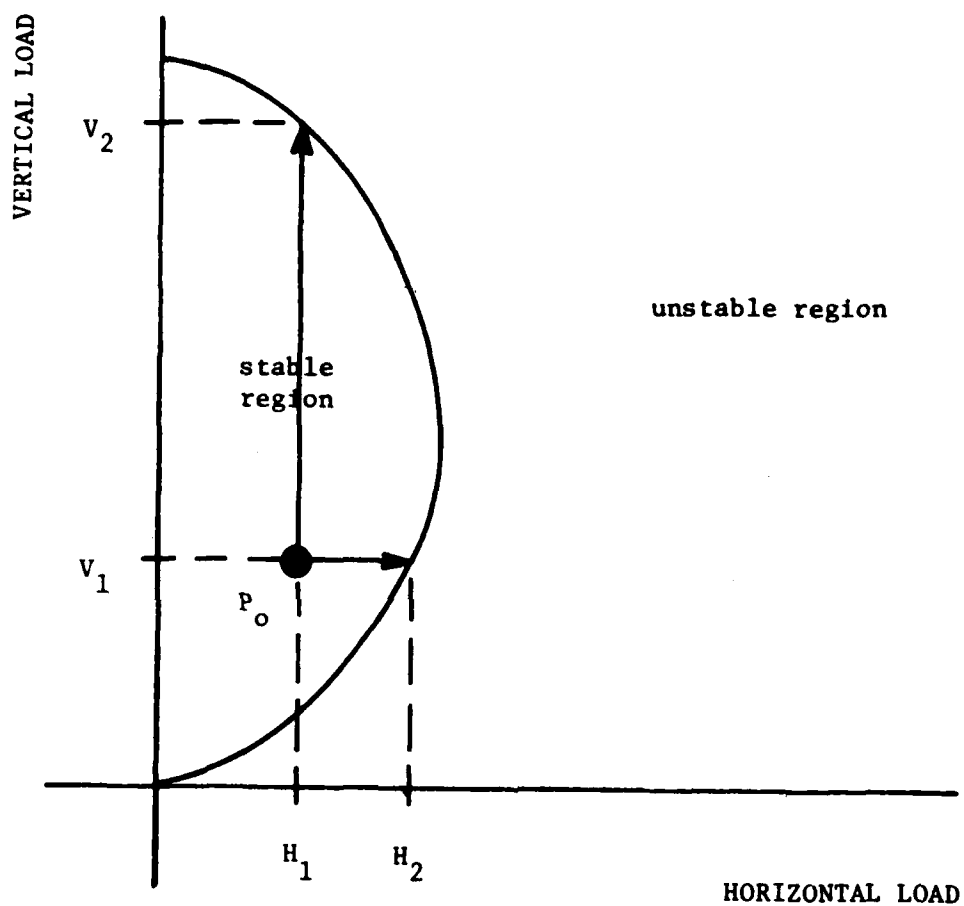
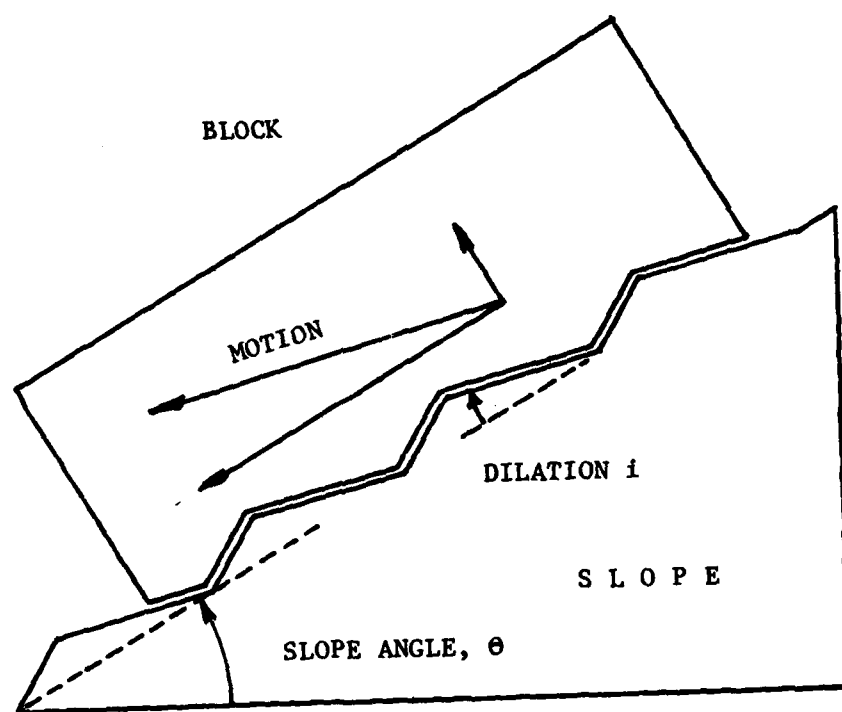Figure 30. Load Path Dependence of Factor of Safety and Reliability Index

Figure 31. Non-invariance of Factor of Safety and Reliability Index
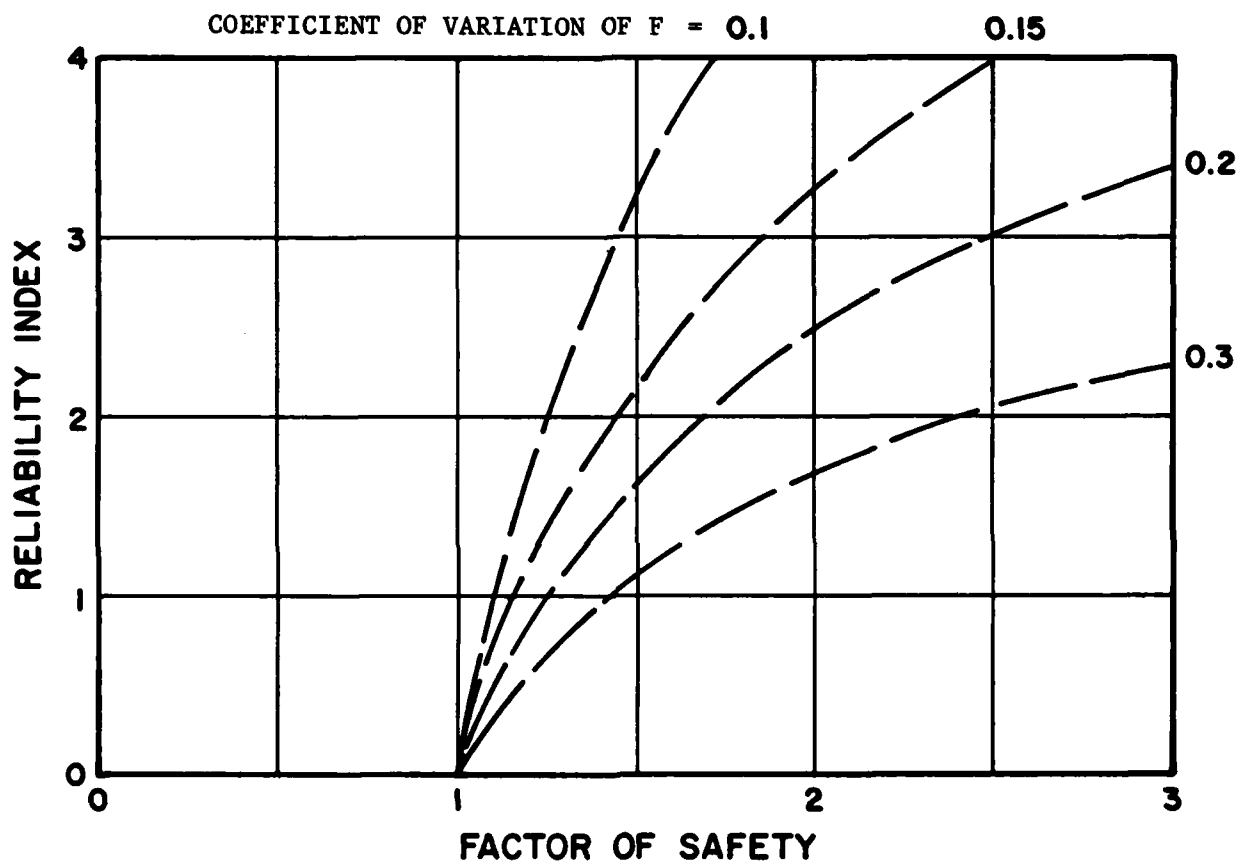
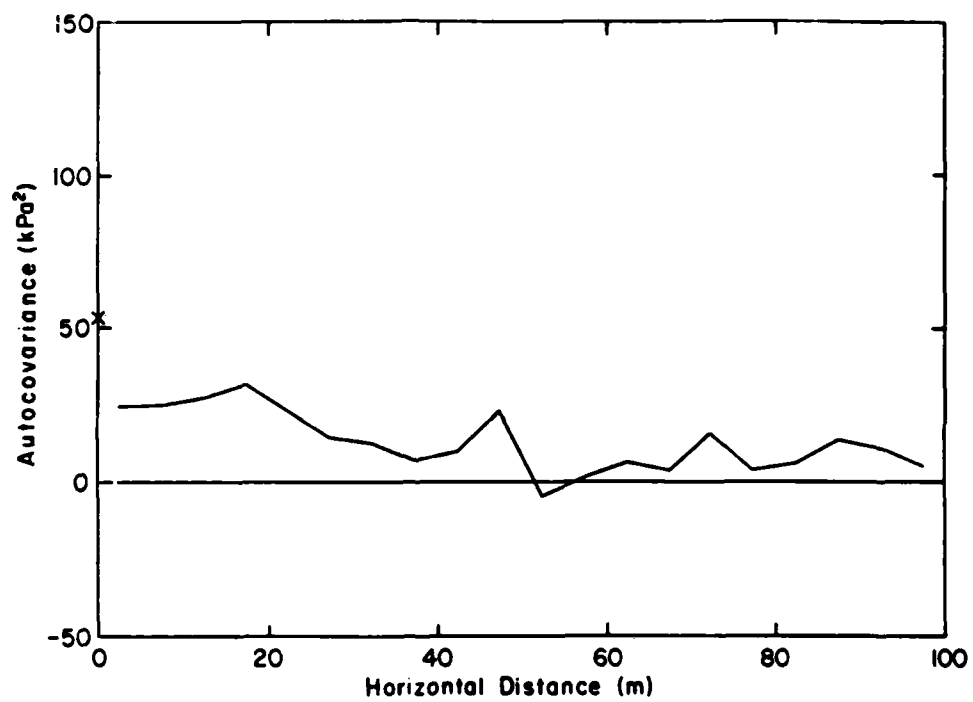Figure 32. Consistent Factors of Safety Based on Equal Reliability
Indices

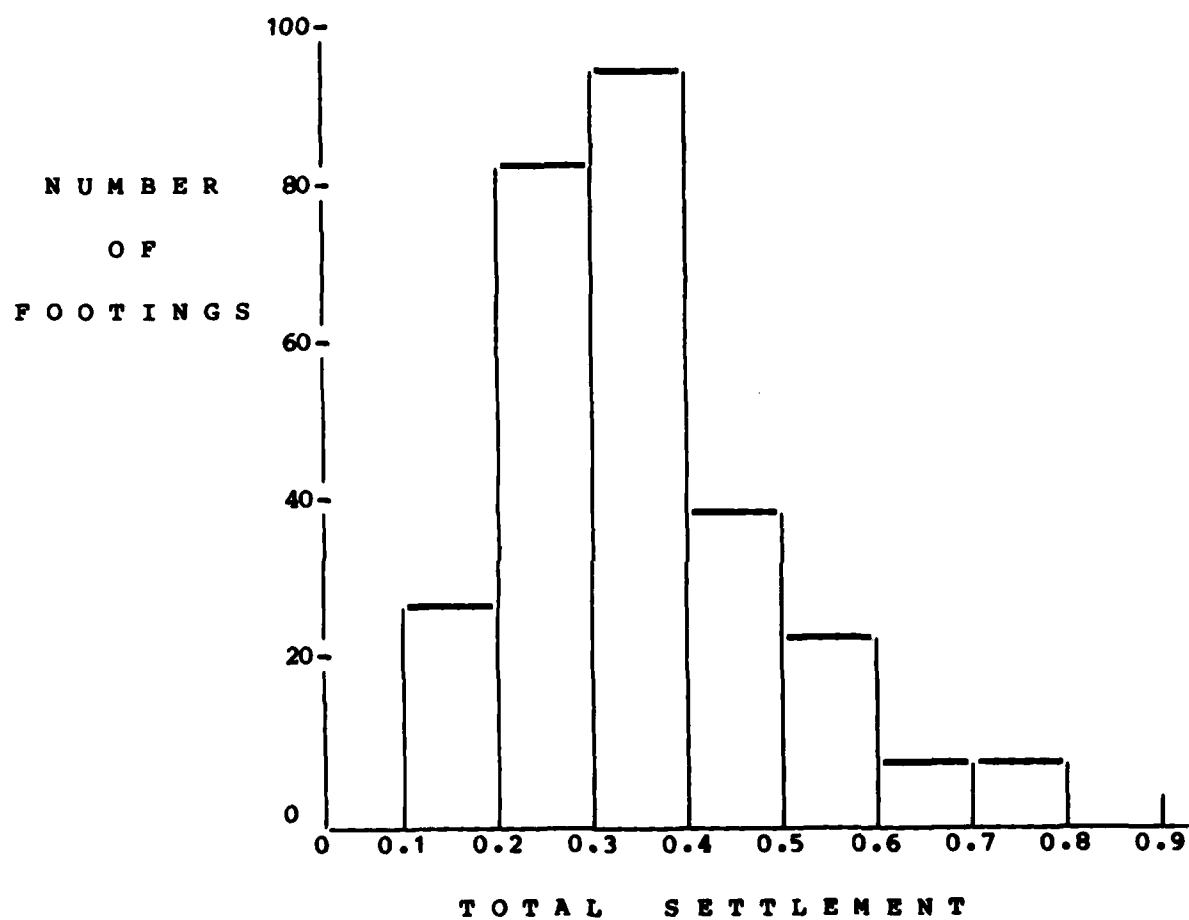Figure 33. Horizontal Autocorrelation for SPT Blow Count Data.

Figure 34. Range of Settlement Predictions Compared with Observations.

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PART V: RISK ANALYSIS

This final part of the report touches briefly on the concept of risk analysis. The intent of Part V is to describe the risk analysis approach to design and to show its relationship to the error analysis procedure presented in previous sections of the report.

Throughout the report, the notion of probabilities has been avoided. To perform an error analysis and to calculate a reliability index β for some design, no assumptions were necessary on probabilities, probability distributions, and so on. The only probabilistic assumption that was needed was that uncertainty could be expressed by a standard deviation. For risk analysis this convenient state of affairs is no longer true. Numerical values of probability are needed to calculate risk costs.

Risk Analysis Defined

Risk analysis is a quantitative approach to balancing design conservatism against the possible consequences and likelihood of adverse facility performance. Of necessity, risk analysis deals with two sides of the design problem. On the one hand it seeks to quantify, usually in numerical probabilities, the likelihhod of adverse performance as a function of design decisions. On the other hand it seeks to quantify, usually in monetary values, the costs associated with adverse performance should a facility not perform as predicted.

Risk analysis attempts to balance the direct costs of constructing a facility against possible costs associated with failure. Typically, the more conservative a design is made, the greater the initial cost but the greater the confidence that the facility will perform satisfactorily.

Conversely, the less conservative, the less the initial cost but the lower the confidence in satisfactory performance. In this example, if a design is too conservative, high initial costs are incurred unnecessarily; but if a design is too unconservative, the potential for failures is unacceptable. Risk analysis intends to provide quantitative guidance in striking a balance between cost and risk.

Practical Implications of the Reliability Index β

Error analysis is a systematic procedure to account for potential errors in engineering calculations. It leads to a best estimate prediction of engineering performance, and to a measure of the confidence that should be placed in the prediction. This measure of confidence is the standard deviation of the possible magnitude of error in the calculation lending to the prediction. The best estimate and standard deviation are combined via Eqn. 41 into a single index, β summarizing the reliability of the prediction in light of a specific design standard or failure criterion. The reliability index β summarizes the confidence one has that the calculation has not erroneously led to a prediction of satisfactory performance.

In a simple way, the question of predicted performance vs real performance can be summarized in the Table 5. The horizontal division shows how the facility actually performs. The vertical division shows what the engineering calculation predicts. The four boxes are the possible combinations of real and predicted performance.

From a practical view, whenever the calculation indicate unsatisfactory performance, either the design or the method of calculation is changed. Thus, the first row in which the calculations indicate satisfactory performance is of

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most concern. In Box 1 at the upper left the calculations correctly indicate satisfactory performance. In Box 2 at the upper right they incorrectly indicate satisfactory performance. The former leads to an acceptable design. The latter leads to failure. For a calculation that indicates satisfactory performance, the reliability index β expresses the confidence of lying in Box 1 rather than Box 2. In the jargon of statistical hypothesis testing, the reliability index β expresses the confidence that a type II error has not been made. A type II error is made when an hypothesis--in this case the hypothesis that the facility will have satisfactory performance--is accepted incorrectly.

Probability of Unsatisfactory Performance

Most risk analysis requires numerical values for the probability of adverse performance of a facility. To calculate a numerical value of probability from the reliability index β some assumption is needed on the probability distribution of possible uncertainties or errors. The probability distribution is a mathematical equation which has the property that areas under its curve within some interval of values equals the probability that the uncertain quantity in question lies within that interval. For example, in Fig. 35 the probability that the uncertain quantity x is less than x_0 equals the crosshatched area under the probability distribution. The probability that x lies in the interval between m_x and $(m_x + s_x)$ equals the stippled area. Since x must lie somewhere, the total area under the probability distribution equals 1.0.

A convenient and often reasonable assumption is that the probability distribution has the bell-shaped form of the Normal distribution (Fig. 36). Empirically, the Normal distribution is often observed to model geotechnical

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data well, and mathematical theory suggests that many types of errors are well represented by Normal distributions.

The difficulty with adopting any particular mathematical function to model the distribution of probability in geotechnical problems is that too few data often exist by which to verify that the chosen distribution provides a good fit, particularly in the outer tails. The Normal distribution is adopted because it is convenient and because it has been found to fit most geotechnical data as well as other common distribution functions do. Actually, the areas under many common distribution functions--for example, the lower part of Figure 36 which shows areas under the Normal curve--are much the same within 2 to 3 standard deviations of the mean, and so within this region little error is made by adopting the Normal distribution. Figure 37 shows the areas under the lower tails (i.e., probabilities of failure) of Normal, logNormal, and Gamma distributions as a function of mean factor of safety (F) and coefficient of variation of F. They are quite similar for Ω's up to 0.2 and probabilities of failure down to 0.01. Nevertheless, to emphasize that the probabilities being calculated are based on a presumed shape for the probability distribution, it is sometimes convenient to refer to them as "nominal probabilities."

Risk Cost

While the concept of direct construction costs is familiar, the concept of risk cost is not. It is generally analogous to the cost of liability and casualty insurance. Since the government self-insures, this principle has not often been considered in the past, yet it is a real cost. In a qualitative sense, the greater the likelihood of unsatisfactory performance and the greater the associated consequences of that performance, the greater the risk cost.

The most common quantitative measure of (economic) risk cost is the product of the probability of unsatisfactory performance times the dollar value of the consequences associated with unsatisfactory performance,

$$C_r = P_f \cdot C_f , \qquad (56)$$

in which C_r = risk cost, P_f = probability of "failure," and C_f = dollar cost associated with failure. For example, if the probability of a shallow footing settling excessively were 0.05 and the cost associated with that settlement were $1000, the risk cost would be C_r = (0.05)($1000), or $50. In principle, reducing that risk through more conservative design would be worth no more than the $50.

The risk cost of Eqn. 56 is a convenient quantitative measure of risk for comparing design alternatives, but it has two important drawbacks. The first is that it does not include non-monetary consequences of inadequate performance, for example, health and safety consequences. Second, it equates a small probability of a large consequence with a moderate probability of a more modest consequence, something many people are not comfortable doing. Each of these issues is outside the scope of the present report. Considerations of non-monetary consequence in risk analysis, particularly loss of life, are discussed by Mishan (1976). The considerations are the same for risk analysis and cost-benefit analysis. Considerations on the question of low-probability high-consequence failures are discussed by Raiffa (1964). These, too, arise in many areas of policy analysis.

Optimal Design

Optimal design means a balancing between direct cost and risk cost. As the conservatism of design increases, as for example by increasing the factor

of safety, direct cost also typically increases. At the same time, however, the probability of unsatisfactory performance and hence the risk cost declines. The total cost, which is the sum of these two,

$$C_{total} = C_{direct} + C_{risk} , \qquad (57)$$

reaches a minimum somewhere along the way (Fig. 38). From a monetary risk viewpoint, this point of least total cost is the optimal design. Plate 8 illustrates the result of a risk assessment of the embankment design of Plate 4.

Value of Information

Risk analysis can also be used to assess the value of information to a geotechnical design. Increased information reduces the uncertainty in a prediction of engineering performance, and by so doing reduces the probability of unsatisfactory behavior and thus risk cost. This is shown schematically in Fig. 38. Increasing the amount of information about site conditions or about measurement or model biases lowers the risk-cost for a given value of the design factor of safety. It does so by reducing statistical error and by reducing calibration errors in testing.

The value of additional information can be calculated by comparing the total cost curves with and without the new information. In a situation where health and safety are not considerations, the difference in optional cost is the maximum amount one should be willing to pay for the new information. Detailed discussion of value-of-information calculations for geotechnical engineering problems is outside the scope of this report. A introductory presentation of general principles is given by Raiffa (1964), and a more

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detailed application to geotechnical data collection for underground

construction is given by Einstein, et al. (1978).

Actual Performance

	ACCEPTABLE	UNACCEPTABLE
ACCEPTABLE	acceptable design	failure
UNACCEPTABLE	<redesign or	reanalyze>

P
r
e
d
i
c
t
e
d

p
e
r
f
o
r
m
a
n
c
e

Table 5. Hypothesis of Adequate Performance.

PLATE 8

SUBJECT: Optimization of factor of safety for embankment stability

PROBLEM: Determine design factor of safety which balances marginal changes in direct cost and in risk cost for the analysis of Plate 4.

SOLUTION: Construction Cost: Based on engineering cost estimates,

$$C_C \approx \ln(1 + 1.146 \, F)$$

in which C_C is direct cost in dollars, and F is the design factor of safety.

Risk Cost: Based on Eqn. 56,

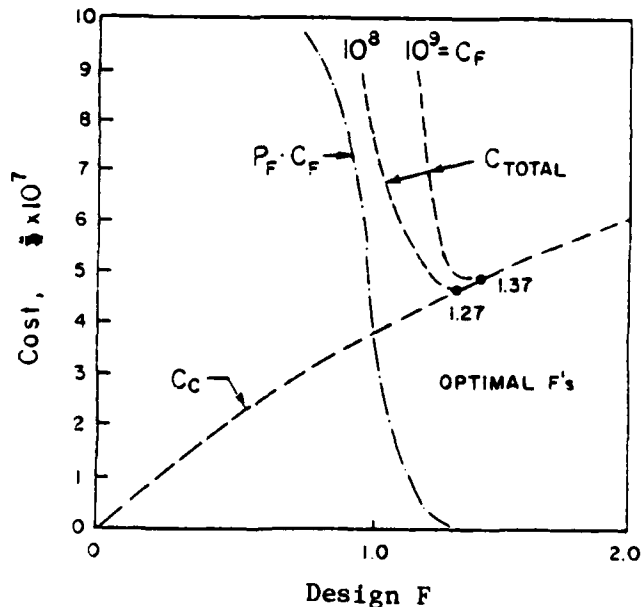$$C_r = P_f \, C_f$$

Assume upper and lower bound costs of

$$C_f^+ = \$10^9$$

$$C_f^- = \$10^9$$

Nominal probabilities of failure based on Fig. 29 and calculations of Plate 4 and extensions to other F's.

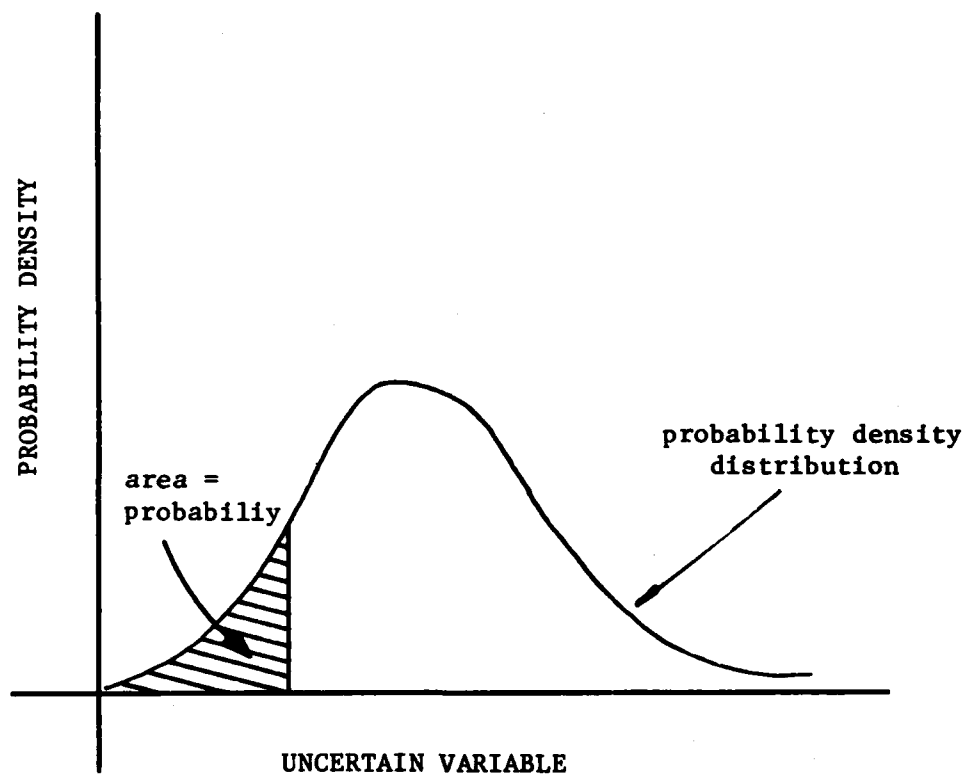Result: Cost model: $C_C = \ell n \, (1 + 1.146 F)$
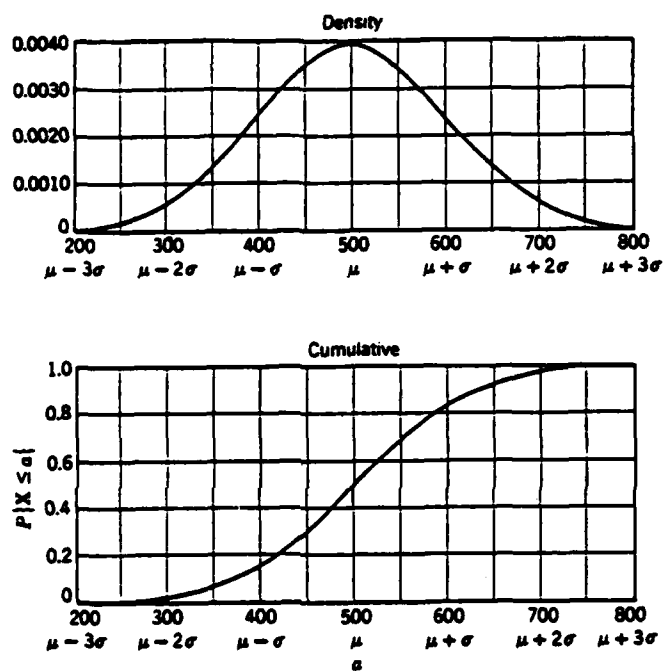
Figure 35. Probability Distribution, Schematic.
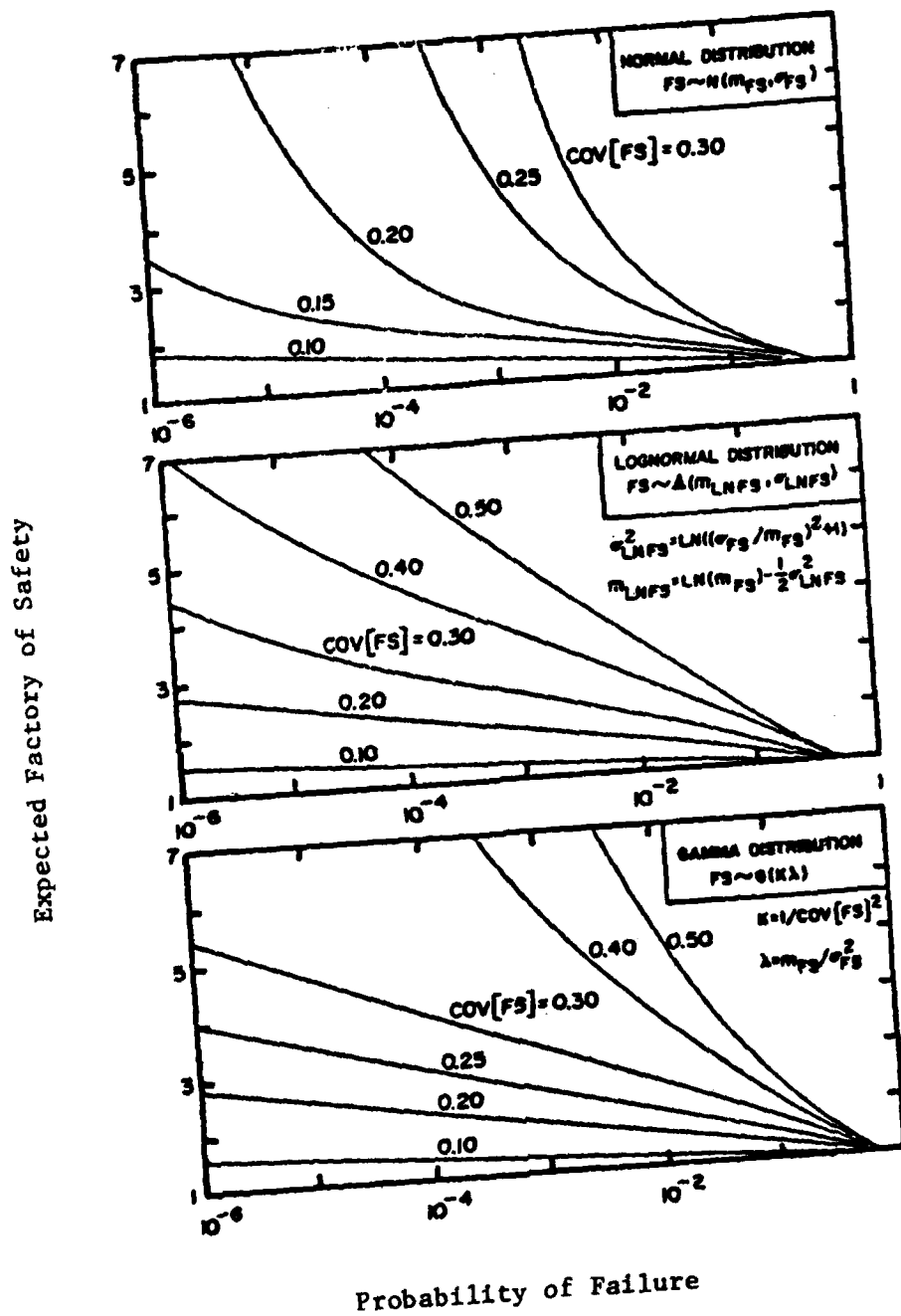
Figure 36. Normal Probability Distribution

Figure 37. Probability of Failure for Various Probability Distributions.
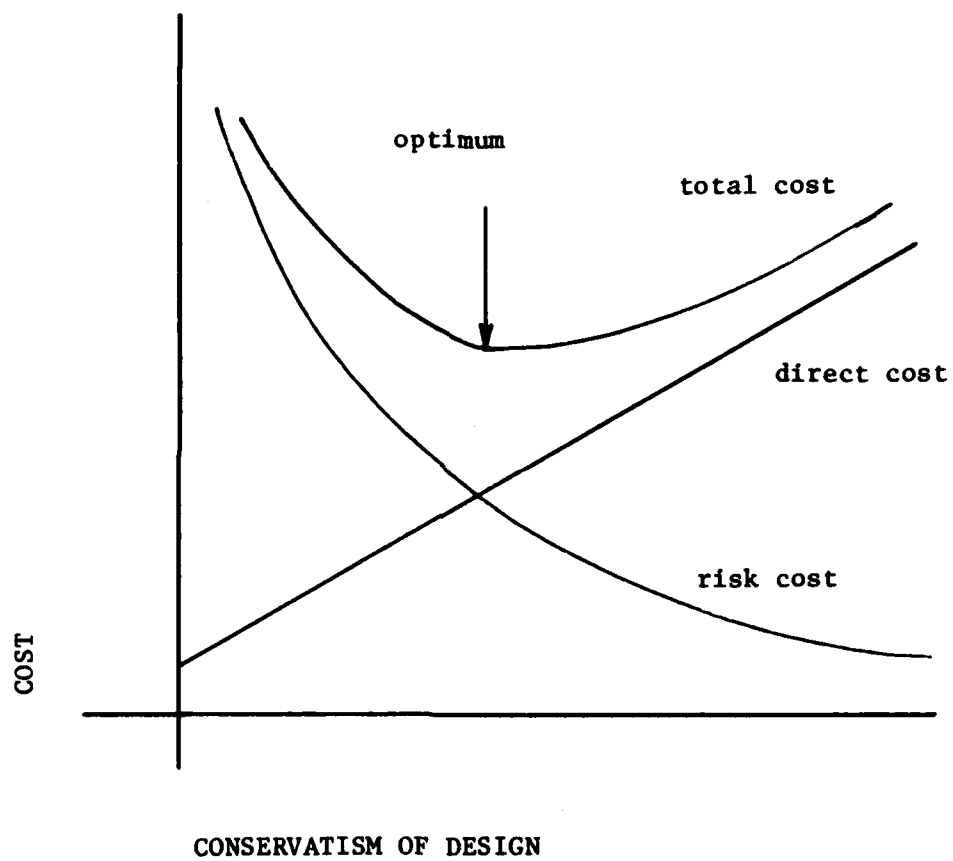
Figure 38. Optimal Design.

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Appendix A -- SYMBOL LIST

a,b = regression coefficients

a_i = constant

B = measurement bias correction coefficient

b = footing width

C_f = cost of failure

C_R = risk cost

$C_x(\delta)$ = autocovariance function for separation distance δ

$C_{x,y}$ = covariance of x and y

$\underline{C}_x$ = covariance matrix

$CR_{\epsilon c}$ = virgin compression ratio

$CR_{\epsilon r}$ = recompression ratio

c_u = undrained strength

d = embedment depth of footing

D,d = geometric properties of scatter graph

e = random measurement error

f_i = cumulative frequency of observation i

E = elastic modulus

F = factor of safety

FV = field vane

$\underline{G}$ = matrix of derivatives with ij^{th} element dy_i/dx_j

$g(x)$ = deterministic function of x

H = horizontal load

H,h = geometric properties of scatter graph

H_i = stratum thickness

h = SHANSEP strength parameter

i = dilation angle

k = counter number

m_x = mean of x

n = number of measurements

L = length

$L[z]$ = likelihood of z

m_v = vertical compression coefficient

N = SPT blow count

N_γ = bearing capacity factor

OCR = overconsolidation ratio

P_{BC} = probability of bearing capacity failure

P_f = probability of failure

P_ρ = probability of excessive settlement

$Pr\{.\}$ = probability of

q = SHANSEP strength parameter

q = applied footing stress

q_{vo} = design stress

q_v = bearing capacity

r_{xy} = correlation coefficient of xy

r_o = autocorrelation distance, $C_.(r_o)=1/e$

R = size effect factor

$R_x(\delta)$ = autocorrelation function over separation distance δ

s_x = standard deviation of x

A1

t = Student's t statistic

t_i = trend

u = residual variation about regression line

V = vertical load

V_x = variance of x

w_x = range of x

x = soil property

$\underline{x}$ = vector of data $x_1, \ldots, x_n$

$x_i,$ = i^{th} measurement of property x, or x at location i

x_{max} = largest value of x

x_{min} = smallest value of x

$x_{0.25}$ = 25th fractile of x

$x_{0.5}$ = 50th fractile of x

$x_{0.75}$ = 75th fractile of x

y = predicted performance variable

y_o = design specification on variable y

z = measured soil property, depth

α = critical probability level

β = reliability index

$\underline{\beta}$ = vector of regression coefficients

γ = soil density

δ = separation distance

δ_o = autocorrelation distance

ϵ = strain

η = point of expansion in Taylor's series

θ = slope angle

μ = Bjerrum's FV correction factor

ν = degrees of freedom

ρ = settlement

σ = stress

σ_{vm}' = maximum past pressure

σ_{vo}' = effective vertical stress

σ_{vf}' = final consolidation stress

ϕ' = effective stress friction angle

Ω_x = coefficient of variation

ω = outlier test statistic

END

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